Two linear transformations each tridiagonal with respect to an eigenbasis of the other

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Abstract

Let $\mathcal{F}$ denote a field, and let V denote a vector space over $\mathcal{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ satisfying both conditions below:

(i) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.

We call such a pair a *Leonard pair* on V. Refining this notion a bit, we introduce the concept of a *Leonard system*. We give a complete classification of Leonard systems. Integral to our proof is the following result. We show that for any Leonard pair A, A^* on V, there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathcal{F}$ such that both

$$
\begin{aligned}
0 &= [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \\
0 &= [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A],
\end{aligned}
$$

where $[r, s]$ means $rs - sr$. The sequence is uniquely determined by the Leonard pair if the dimension of V is at least 4. We conclude by showing how Leonard systems correspond to q-Racah and related polynomials from the Askey scheme.

Keywords: q-Racah polynomial, Askey scheme, subconstituent algebra, Terwilliger algebra, Askey-Wilson algebra, Dolan-Grady relations, quadratic algebra, Serre relations

CONTENTS

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1 Introduction

Throughout this paper, $\mathcal{F}$ will denote an arbitrary field.

We begin with the following situation in linear algebra.

Definition 1.1 *Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. By a Leonard pair on V, we mean an ordered pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i), (ii) below.*

(i) There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

(A tridiagonal matrix is said to be irreducible whenever all entries immediately above and below the main diagonal are nonzero).

Note 1.2 *According to a common notational convention, for a linear transformation A the conjugate-transpose of A is denoted A^*. We emphasize we are not using this convention. In a Leonard pair (A, A^*), the linear transformations A and A^* are arbitrary subject to (i), (ii) above.*

Our use of the name "Leonard pair" is motivated by a connection to a theorem of Leonard [27], [5, p260] involving the q-Racah and related polynomials of the Askey scheme [25]. For more information on this, we refer the reader to Section 15.

Here is an example of a Leonard pair. Set $V = \mathcal{F}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view A and A^* as linear transformations from V to V. We assume the characteristic of $\mathcal{F}$ is not 2 or 3, to ensure A is irreducible. Then (A, A^*) is a Leonard pair on V. Indeed, condition (i) in Definition 1.1 is satisfied by the basis for V consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix P such that $P^{-1}AP$ is diagonal, and such that $P^{-1}A^*P$ is irreducible tridiagonal. Put

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, where I denotes the identity, so P^{-1} exists. Also by matrix multiplication,

$$AP = PA^*. \tag{1}$$

Apparently $P^{-1}AP$ equals A^*, and is therefore diagonal. By (1), and since P^{-1} is a scalar multiple of P, we find $P^{-1}A^*P$ equals A, and is therefore irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for V consisting of the columns of P.

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract object, which we call a *Leonard system*. In order to define this, we first make an observation about Leonard pairs.

Lemma 1.3 *With reference to Definition 1.1, let (A, A^*) denote a Leonard pair on V. Then the eigenvalues of A are distinct and contained in $\mathcal{F}$. Moreover, the eigenvalues of A^* are distinct and contained in $\mathcal{F}$.*

Proof: Concerning A, recall by Definition 1.1(ii) that there exists a basis for V consisting of eigenvectors for A. Consequently the eigenvalues of A are all in $\mathcal{F}$, and the minimal polynomial of A has no repeated roots. To show the eigenvalues of A are distinct, we show the minimal polynomial of A has degree equal to $\dim V$. By Definition 1.1(i), there exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal. Denote this matrix by B. On one hand, A and B have the same minimal polynomial. On the other hand, using the tridiagonal shape of B, we find $I, B, B^2, \ldots, B^d$ are linearly independent, where $d = \dim V - 1$, so the minimal polynomial of B has degree $d + 1 = \dim V$. We conclude the mininimal polynomial of A has degree equal to $\dim V$, so the eigenvalues of A are distinct. We have now obtained our assertions about A, and the case of A^* is similar.
$\Box$

To prepare for our definition of a Leonard system, we recall a few concepts from elementary linear algebra. Let d denote a nonnegative integer, and let $\mathrm{Mat}_{d+1}(\mathcal{F})$ denote the $\mathcal{F}$-algebra consisting of all $d+1$ by $d+1$ matrices with entries in $\mathcal{F}$. We index the rows and columns by $0, 1, \ldots, d$. For the rest of this paper, $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathcal{F})$. Let A denote an element of $\mathcal{A}$. By an *eigenvalue* of A, we mean a root of the minimal polynomial of A. The eigenvalues of A are contained in the algebraic closure of $\mathcal{F}$. The element A will be called *multiplicity-free* whenever it has $d + 1$ distinct eigenvalues, all of which are in $\mathcal{F}$. Let A denote a multiplicity-free element of $\mathcal{A}$. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leq i \leq d$ put

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of $\mathcal{A}$. By elementary linear algebra,

$$AE_i = E_i A = \theta_i E_i \qquad (0 \leq i \leq d), \tag{2}$$

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq d), \tag{3}$$

$$\sum_{i=0}^{d} E_i = I. \tag{4}$$

From this, one finds $E_0, E_1, \ldots, E_d$ is a basis for the subalgebra of $\mathcal{A}$ generated by A. We refer to E_i as the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Let V denote the irreducible left $\mathcal{A}$-module. Then

$$V = E_0V + E_1V + \cdots + E_dV \qquad \text{(direct sum)}. \tag{5}$$

For $0 \leq i \leq d$, E_iV is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

Definition 1.4 *Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard system in $\mathcal{A}$, we mean a sequence*

$$\Phi = (A;\, E_0,\, E_1,\, \ldots,\, E_d;\, A^*;\, E_0^*,\, E_1^*,\, \ldots,\, E_d^*) \tag{6}$$

that satisfies (i)–(v) below.

(i) A, A^* *are both multiplicity-free elements in* $\mathcal{A}$.

(ii) $E_0, E_1, \ldots, E_d$ *is an ordering of the primitive idempotents of* A.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ *is an ordering of the primitive idempotents of* A^*.

(iv) $E_iA^*E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i, j \leq d).$

(v) $E_i^*AE_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i, j \leq d).$

We refer to d as the diameter of Φ, and say Φ is over $\mathcal{F}$. We sometimes write $\mathcal{A} = \mathcal{A}(\Phi)$, $\mathcal{F} = \mathcal{F}(\Phi)$. For notational convenience, we set $E_{-1} = 0$, $E_{d+1} = 0$, $E_{-1}^* = 0$, $E_{d+1}^* = 0$.

To see the connection between Leonard pairs and Leonard systems, observe conditions (ii), (iv) above assert that with respect to an appropriate basis consisting of eigenvectors for A, the matrix representing A^* is irreducible tridiagonal. Similarily, conditions (iii), (v) assert that with respect to an appropriate basis consisting of eigenvectors for A^*, the matrix representing A is irreducible tridiagonal.

A little later in this introduction, we will state our main results, which are Theorems 1.9, 1.11, and 1.12. For now, we mention some of the concepts that get used.

Let Φ denote the Leonard system in (6), and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathcal{F}$-algebras. We write

$$\Phi^\sigma := (A^\sigma; E_0^\sigma, E_1^\sigma, \ldots, E_d^\sigma; A^{*\sigma}; E_0^{*\sigma}, E_1^{*\sigma}, \ldots, E_d^{*\sigma}), \tag{7}$$

and observe Φ^σ is a Leonard system in $\mathcal{A}'$.

Definition 1.5 *Let Φ and Φ' denote Leonard systems over $\mathcal{F}$. By an isomorphism of Leonard systems from Φ to Φ', we mean an isomorphism of $\mathcal{F}$-algebras $\sigma : \mathcal{A}(\Phi) \to \mathcal{A}(\Phi')$ such that $\Phi^\sigma = \Phi'$. The Leonard systems Φ, Φ' are said to be isomorphic whenever there exists an isomorphism of Leonard systems from Φ to Φ'.*

A given Leonard system can be modified in several ways to get a new Leonard system. For instance, let Φ denote the Leonard system in (6), and let α, α^*, β, β^* denote scalars in $\mathcal{F}$ such that $\alpha \neq 0$, $\alpha^* \neq 0$. Then

$$(\alpha A + \beta I; E_0, E_1, \ldots, E_d; \alpha^* A^* + \beta^* I; E_0^*, E_1^*, \ldots, E_d^*)$$

is a Leonard system in $\mathcal{A}$. Also,

$$\begin{aligned} \Phi^* &:= (A^*; E_0^*, E_1^*, \ldots, E_d^*; A; E_0, E_1, \ldots, E_d), && (8) \\ \Phi^\downarrow &:= (A; E_0, E_1, \ldots, E_d; A^*; E_d^*, E_{d-1}^*, \ldots, E_0^*), && (9) \\ \Phi^\Downarrow &:= (A; E_d, E_{d-1}, \ldots, E_0; A^*; E_0^*, E_1^*, \ldots, E_d^*) && (10) \end{aligned}$$

are Leonard systems in $\mathcal{A}$. We refer to Φ^* (resp. $\Phi^{\downarrow}$) (resp. $\Phi^{\Downarrow}$) as the *dual* (resp. *first inversion*) (resp. *second inversion*) of Φ. Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{11}$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \Downarrow\downarrow . \tag{12}$$

The group generated by symbols $*, \downarrow, \Downarrow$ subject to the relations (11), (12) is the dihedral group D_4. We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action. Assuming $d \geq 1$ to avoid trivialities, the relatives of Φ are as follows:

name	relative
Φ	$(A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$
$\Phi^{\downarrow}$	$(A; E_0, E_1, \ldots, E_d; A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0)$
$\Phi^{\Downarrow}$	$(A; E_d, E_{d-1}, \ldots, E_0; A^*; E^*_0, E^*_1, \ldots, E^*_d)$
$\Phi^{\downarrow\Downarrow}$	$(A; E_d, E_{d-1}, \ldots, E_0; A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0)$
Φ^*	$(A^*; E^*_0, E^*_1, \ldots, E^*_d; A; E_0, E_1, \ldots, E_d)$
$\Phi^{\downarrow *}$	$(A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0; A; E_0, E_1, \ldots, E_d)$
$\Phi^{\Downarrow *}$	$(A^*; E^*_0, E^*_1, \ldots, E^*_d; A; E_d, E_{d-1}, \ldots, E_0)$
$\Phi^{\downarrow\Downarrow *}$	$(A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0; A; E_d, E_{d-1}, \ldots, E_0)$

We remark there may be some isomorphisms among the above Leonard systems.
In view of our above comments, when we discuss a Leonard system, we are often not interested in the orderings of the primitive idempotents, we just care how the A and A^* elements interact. This brings us back to the notion of a Leonard pair.

Definition 1.6 *Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let $\mathcal{A}$ denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. By a Leonard pair in $\mathcal{A}$, we mean an ordered pair (A, A^*) satisfying both (i), (ii) below.*

(i) A and A^* are both multiplicity-free elements in $\mathcal{A}$.

(ii) There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of A, and there exists an ordering $E^*_0, E^*_1, \ldots, E^*_d$ of the primitive idempotents of A^*, such that

$$(A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$$

is a Leonard system.

We refer to d as the diameter of the pair, and say the pair is over $\mathcal{F}$.

Let Φ denote the Leonard system in (6). Apparently the pair (A, A^*) from that line is a Leonard pair in $\mathcal{A}$, which we say is *associated* with Φ. Let (A, A^*) denote a Leonard pair in $\mathcal{A}$. Then (A^*, A) is a Leonard pair in $\mathcal{A}$, which we call the *dual* of (A, A^*). It is routine to show two Leonard systems are relatives if and only if their associated Leonard pairs are equal or dual.

In the following lemma, we make explicit the connection between the notions of Leonard pair that appear in Definition 1.1 and Definition 1.6. The proof is routine and left to the reader.

Lemma 1.7 *Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. Let $End(V)$ denote the $\mathcal{F}$-algebra consisting of all linear transformations from V to V, and recall $End(V)$ is $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$, where $d+1 = dim(V)$. Then for all A and A^* in $End(V)$, the following are equivalent.*

(i) (A, A^*) *is a Leonard pair on* V*, in the sense of Definition 1.1.*

(ii) (A, A^*) *is a Leonard pair in End(V), in the sense of Definition 1.6.*

We now introduce four sequences of parameters that we will use to describe a given Leonard system. The first two sequences are given in the following definition.

Definition 1.8 *Let* Φ *denote the Leonard system in (6). For* $0 \le i \le d$*, we let* θ_i *(resp.* θ^*_i*) denote the eigenvalue of* A *(resp.* A^**) associated with* E_i *(resp.* E^*_i*). We refer to* $\theta_0, \theta_1, \ldots, \theta_d$ *as the eigenvalue sequence of* Φ*. We refer to* $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ *as the dual eigenvalue sequence of* Φ*.*

There are two more parameter sequences of interest to us. Let Φ denote the Leonard system in (6). As we will show in Theorem 3.2, there exists an isomorphism of $\mathcal{F}$-algebras $\flat : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathcal{F})$ and there exists scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathcal{F}$ such that

$$A^\flat = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\flat} = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix},$$

where the θ_i, θ^*_i are from Definition 1.8. The sequence $\flat, \varphi_1, \varphi_2, \ldots, \varphi_d$ is uniquely determined by Φ. We refer to $\varphi_1, \varphi_2, \ldots, \varphi_d$ as the *φ-sequence* of Φ. We let $\phi_1, \phi_2, \ldots, \phi_d$ denote the φ-sequence of $\Phi^\Downarrow$, and call this the *ϕ-sequence* of Φ.

The central result of this paper is the following classification of Leonard systems.

Theorem 1.9 *Let* d *denote a nonnegative integer, let* $\mathcal{F}$ *denote a field, and let*

$$\theta_0, \theta_1, \ldots, \theta_d; \qquad \theta^*_0, \theta^*_1, \ldots, \theta^*_d; \tag{13}$$

$$\varphi_1, \varphi_2, \ldots, \varphi_d; \qquad \phi_1, \phi_2, \ldots, \phi_d \tag{14}$$

denote scalars in $\mathcal{F}$*. Then there exists a Leonard system* Φ *over* $\mathcal{F}$ *with eigenvalue sequence* $\theta_0, \theta_1, \ldots, \theta_d$*, dual eigenvalue sequence* $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$*,* φ*-sequence* $\varphi_1, \varphi_2, \ldots, \varphi_d$*, and* ϕ*-sequence* $\phi_1, \phi_2, \ldots, \phi_d$ *if and only if (i)–(v) hold below.*

(i) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad\qquad (1 \le i \le d),$

(ii) $\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad$ *if* $\; i \neq j, \qquad\qquad (0 \le i, j \le d),$

(iii) $$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \; + \; (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \le i \le d),$$

(iv) $$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \; + \; (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \le i \le d),$$

(v) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{15}$$

are equal and independent of i*, for* $\; 2 \le i \le d-1$.

Moreover, if (i)–(v) hold above then Φ is unique up to isomorphism of Leonard systems.

The proof of Theorem 1.9 appears in Section 14.

We view Theorem 1.9 as a linear algebraic version of a theorem of Leonard [27], [5, p260]. This is discussed in Section 15.

We have found all solutions to Theorem 1.9(i)–(v) in parametric form. We will present these in a future paper, and for now display only the "most general" solution. It is

$$\begin{aligned} \theta_i &= \theta_0 + h(1-q^i)(1-sq^{i+1})/q^i, \\ \theta^*_i &= \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})/q^i \end{aligned}$$

for $0 \le i \le d$, and

$$\begin{aligned} \varphi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i), \\ \phi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^* \end{aligned}$$

for $1 \le i \le d$, where $q, h, h^*, r_1, r_2, s, s^*$ are scalars in the algebraic closure of $\mathcal{F}$ such that $r_1r_2 = ss^*q^{d+1}$. For this solution the common value of (15) equals $q + q^{-1} + 1$.

One nice feature of the parameter sequences (13), (14) is that they are modified in a simple way as one passes from a given Leonard system to a relative. To describe how this works, we use the following notational convention.

Definition 1.10 *Let Φ denote a Leonard system. For any element g of the group D_4, and for any object f we associate with Φ, we let f^g denote the corresponding object associated with the Leonard system $\Phi^{g^{-1}}$. We have been using this convention all along; an example is $\theta^*_i(\Phi) = \theta_i(\Phi^*)$.*

Theorem 1.11 *Let Φ denote a Leonard system, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$. Then for all $g \in D_4$, the scalars θ^g_i, θ^{*g}_i, φ^g_i, ϕ^g_i are as follows.*

g	θ^g_i	θ^{*g}_i	φ^g_i	ϕ^g_i
1	θ_i	θ^*_i	φ_i	ϕ_i
$\downarrow$	θ_i	θ^*_{d-i}	ϕ_{d-i+1}	φ_{d-i+1}
$\Downarrow$	θ_{d-i}	θ^*_i	ϕ_i	φ_i
$\downarrow\Downarrow$	θ_{d-i}	θ^*_{d-i}	φ_{d-i+1}	ϕ_{d-i+1}
$*$	θ^*_i	θ_i	φ_i	ϕ_{d-i+1}
$\downarrow *$	θ^*_i	θ_{d-i}	ϕ_i	φ_{d-i+1}
$\Downarrow *$	θ^*_{d-i}	θ_i	ϕ_{d-i+1}	φ_i
$\downarrow\Downarrow *$	θ^*_{d-i}	θ_{d-i}	φ_{d-i+1}	ϕ_i

The proof of Theorem 1.11 appears in Section 6.

We show the elements of a Leonard pair satisfy the following relations.

Theorem 1.12 *Let $\mathcal{F}$ denote a field, and let (A, A^*) denote a Leonard pair over $\mathcal{F}$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathcal{F}$ such that both*

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \tag{16}$$
$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A], \tag{17}$$

where $[r,s]$ means $rs-sr$. The sequence is uniquely determined by the Leonard pair if the diameter is at least 3.

The proof of Theorem 1.12 appears at the end of Section 12.

The relations (16), (17) previously appeared in [34]. In that paper, the author considers a combinatorial object called a P- and Q-polynomial association scheme [5], [6], [28], [31], [35]. He shows that for these schemes the adjacency matrix A and a certain diagonal matrix A^* satisfy (16), (17). In this context the algebra generated by A and A^* is known as the subconstituent algebra or the Terwilliger algebra [7], [10], [11], [12], [13], [14], [18], [24], [30], [32], [33].

A special case of (16), (17) occurs in the context of quantum groups. Setting $\beta = q^2 + q^{-2}$, $\gamma = 0$, $\gamma^* = 0$, $\varrho = 0$, $\varrho^* = 0$ in (16), (17), one obtains

$$0 = A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3, \tag{18}$$
$$0 = A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3}, \tag{19}$$

where

$$[3]_q := \frac{q^3 - q^{-3}}{q - q^{-1}}.$$

The equations (18), (19) are known as the *q-Serre relations*, and are among the defining relations for the quantum affine algebra $U_q(\widehat{sl}_2)$ [8], [9].

A special case of (16), (17) has come up in the context of exactly solvable models in statistical mechanics. Setting $\beta = 2, \gamma = 0, \gamma^* = 0, \varrho = 16, \varrho^* = 16$ in (16), (17), one obtains

$$[A,[A,[A,A^*]]] = 16[A,A^*], \tag{20}$$
$$[A^*,[A^*,[A^*,A]]] = 16[A^*,A]. \tag{21}$$

The equations (20), (21) are known as the Dolan–Grady relations [1], [15], [17], [36]. We remark the Lie algebra over $\mathbb{C}$ generated by two symbols A, A^* subject to (20), (21) (where we interpret [,] as the Lie bracket) is infinite dimensional and is known as the Onsager algebra [16], [29]. The author would like to thank Anatol N. Kirillov for pointing out the connection to statistical mechanics.

We mention the relations (16), (17) are satisfied by the generators of both the classical and quantum "Quadratic Askey-Wilson algebra" introduced by Granovskii, Lutzenko, and Zhedanov [20]. See [19], [21], [22], [23], [37], [38], [39] for more information on this algebra.

Given a field $\mathcal{F}$, and given scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathcal{F}$, it is natural in light of our above comments to consider the associative $\mathcal{F}$-algebra generated by two symbols A, A^* subject to the relations (16), (17). It appears to be an open problem to give a basis for this algebra, and to describe all its irreducible representations.

2 Some preliminaries

We now turn to the business of proving the results that we displayed in the Introduction. This will take most of the paper, up through the end of Section 14. We begin with some simple observations.

Definition 2.1 *In this section, d will denote a nonnegative integer, $\mathcal{F}$ will denote a field, and $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. We let V denote the irreducible left $\mathcal{A}$-module. We let A and A^* denote multiplicity-free elements of $\mathcal{A}$. We let $E_0, E_1, \dots E_d$ denote an ordering of the primitive idempotents of A, and we let $E^*_0, E^*_1, \dots E^*_d$ denote an ordering of the primitive idempotents of A^*. For $0 \leq i \leq d$, we let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E^*_i).*

Definition 2.2 *With reference to Definition 2.1, by an (A, A^*)-module, we mean a subspace W of V such that $AW \subseteq W$ and $A^*W \subseteq W$. Let W denote an (A, A^*)-module. We say W is irreducible whenever $W \neq 0$, and W contains no (A, A^*)-modules other than 0 and W.*

With reference to Definition 2.1, let W denote an (A, A^*)-module. Since $AW \subseteq W$, we find $W = \sum_{i \in S} E_iV$, where S is an appropriate subset of $\{0, 1, \dots, d\}$. We now consider which subsets S can occur.

Lemma 2.3 *With reference to Definition 2.1, let S denote a subset of $\{0, 1, \dots, d\}$, and put $W = \sum_{i \in S} E_iV$. Then the following are equivalent.*

(i) W is an (A, A^*)-module.

(ii) $E_iA^*E_j = 0$ for all $j \in S$ and for all $i \in \{0, 1, \dots, d\} \backslash S$.

Proof: $(i) \rightarrow (ii)$ Let the integers i, j be given. Observe $E_jV \subseteq W$ since $j \in S$, and $A^*W \subseteq W$, so $A^*E_jV \subseteq W$. Observe $E_iW = 0$ since $i \notin S$, so $E_iA^*E_jV = 0$. It follows $E_iA^*E_j = 0$.
$(ii) \rightarrow (i)$ Recall E_iV is an eigenspace of A for $0 \leq i \leq d$, so $AW \subseteq W$. We now show $A^*W \subseteq W$. For notational convenience set $J = \sum_{i \in S} E_i$. Let $\overline{S}$ denote the complement of S in $\{0, 1, \dots, d\}$, and set $K = \sum_{i \in \overline{S}} E_i$. Then $J + K = I$ and $KA^*J = 0$. Combining these, we find $A^*J = JA^*J$. Applying this to V, and observing $JV = W$, we routinely find $A^*W \subseteq W$.
$\Box$

Interchanging the roles of A and A^* in the previous lemma, we immediately obtain the following result.

Lemma 2.4 *With reference to Definition 2.1, let S^* denote a subset of $\{0, 1, \dots, d\}$, and put $W = \sum_{i \in S^*} E^*_iV$. Then the following are equivalent.*

(i) W is an (A, A^*)-module.

(ii) $E^*_iAE^*_j = 0$ for all $j \in S^*$ and for all $i \in \{0, 1, \dots, d\} \backslash S^*$.

The following constants will be of use to us.

Definition 2.5 *With reference to Definition 2.1, we define*

$$a_i = tr\, AE^*_i, \qquad a^*_i = tr\, A^*E_i, \qquad (0 \leq i \leq d), \tag{22}$$

where tr means trace.

Lemma 2.6 *With reference to Definition 2.1 and Definition 2.5,*

$$\theta_0 + \theta_1 + \cdots + \theta_d = a_0 + a_1 + \cdots + a_d, \tag{23}$$

$$\theta^*_0 + \theta^*_1 + \cdots + \theta^*_d = a^*_0 + a^*_1 + \cdots + a^*_d. \tag{24}$$

Proof: To get (23), take the trace of both sides in the equation $A = A \sum_{i=0}^{d} E^*_i$, and evaluate the result using the left equation in (22). Line (24) is similarily obtained.
$\Box$

3 The split canonical form

Definition 3.1 *In this section, d will denote a nonnegative integer, $\mathcal{F}$ will denote a field, and $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to $Mat_{d+1}(\mathcal{F})$. We let V denote the irreducible left $\mathcal{A}$-module. We let*

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$$

denote a Leonard system in $\mathcal{A}$, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$.

With reference to Definition 3.1, when studying Φ, it is tempting to represent one of A and A^* by a diagonal matrix, and the other by an irreducible tridiagonal matrix. This approach has some merit, but we are going to do something else. Our goal in this section is to prove the following theorem.

Theorem 3.2 *With reference to Definition 3.1, there exists an isomorphism of $\mathcal{F}$-algebras $\flat : \mathcal{A} \rightarrow Mat_{d+1}(\mathcal{F})$ and there exists scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathcal{F}$ such that*

$$A^\flat = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^{*\flat} = \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}. \tag{25}$$

The sequence $\flat, \varphi_1, \varphi_2, \ldots, \varphi_d$ is uniquely determined by Φ. Moreover $\varphi_i \neq 0$ for $1 \leq i \leq d$.

We begin with an irreducibility result.

Lemma 3.3 *With reference to Definition 3.1, The module V is irreducible as an (A, A^*)-module.*

Proof: Let W denote a nonzero (A, A^*)-module in V. We show $W = V$. Since $AW \subseteq W$, there exists a subset S of $\{0, 1, \ldots, d\}$ such that $W = \sum_{h \in S} E_h V$. From Lemma 2.3(ii) and the definition of a Leonard system, we find

$$j \in S \quad \text{and} \quad |i - j| = 1 \quad \rightarrow \quad i \in S, \tag{26}$$

for $0 \leq i, j \leq d$. Observe $S \neq \emptyset$ since $W \neq 0$. Combining this with (26), we find $S = \{0, 1, \ldots, d\}$, so $W = V$.
□

Definition 3.4 *With reference to Definition 3.1, we set*

$$V_{ij} = \left(\sum_{h=0}^{i} E_h^* V \right) \cap \left(\sum_{k=j}^{d} E_k V \right) \tag{27}$$

for all integers i, j. We interpret the sum on the left in (27) to be 0 (resp. V) if $i < 0$ (resp. $i > d$). Similarily, we interpret the sum on the right in (27) to be V (resp. 0) if $j < 0$ (resp. $j > d$).

Lemma 3.5 *With reference to Definition 3.1 and Definition 3.4, we have*

(i) $V_{i0} = E^*_0V + E^*_1V + \cdots + E^*_iV \qquad (0 \le i \le d)$,

(ii) $V_{dj} = E_jV + E_{j+1}V + \cdots + E_dV \qquad (0 \le j \le d)$.

Proof: To get (i), set $j = 0$ in (27), and apply (5). Line (ii) is similarily obtained.
□

Lemma 3.6 *With reference to Definition 3.1 and Definition 3.4, the following (i)–(iv) hold for* $0 \le i, j \le d$.

(i) $(A - \theta_jI)V_{ij} \subseteq V_{i+1,j+1}$,

(ii) $AV_{ij} \subseteq V_{ij} + V_{i+1,j+1}$,

(iii) $(A^* - \theta^*_iI)V_{ij} \subseteq V_{i-1,j-1}$,

(iv) $A^*V_{ij} \subseteq V_{ij} + V_{i-1,j-1}$.

Proof: (i) Recall $E^*_rAE^*_h = 0$ for $r - h > 1$, so

$$(A - \theta_jI)\sum_{h=0}^{i} E^*_hV \subseteq \sum_{h=0}^{i+1} E^*_hV. \tag{28}$$

Using (2), we obtain

$$(A - \theta_jI)\sum_{h=j}^{d} E_hV \subseteq \sum_{h=j+1}^{d} E_hV. \tag{29}$$

Evaluating $(A - \theta_jI)V_{ij}$ using (27), (28), (29), we routinely find it is contained in $V_{i+1,j+1}$.
(ii) Immediate from (i) above.
(iii),(iv) Apply (i), (ii) above to $\Phi^{\downarrow\Downarrow *}$
□

Lemma 3.7 *With reference to Definition 3.1 and Definition 3.4, we have*

$$V_{ij} = 0 \quad \textit{if} \quad i < j \qquad (0 \le i, j \le d). \tag{30}$$

Proof: We show the sum

$$V_{0r} + V_{1,r+1} + \cdots + V_{d-r,d} \tag{31}$$

equals 0 for $0 < r \le d$. Let r be given, and let W denote the sum in (31). Applying Lemma 3.6(ii),(iv), we find $AW \subseteq W$ and $A^*W \subseteq W$, so W is an (A, A^*)-module. Applying Lemma 3.3, we find $W = 0$ or $W = V$. By Definition 3.4, each term in (31) is contained in

$$E_rV + E_{r+1}V + \cdots + E_dV, \tag{32}$$

so W is contained in (32). Apparently $W \ne V$, so $W = 0$. We have now shown (31) is zero for $0 < r \le d$, and (30) follows.
□

Lemma 3.8 *With reference to Definition 3.1 and Definition 3.4, we have*

$$dim\, V_{ii} = 1 \qquad (0 \le i \le d) \tag{33}$$

and

$$V = V_{00} + V_{11} + \cdots + V_{dd} \qquad (\textit{direct sum}). \tag{34}$$

Proof: We first show (34). Let W denote the sum on the right in (34). Observe $AW \subseteq W$ by Lemma 3.6(ii), and $A^*W \subseteq W$ by Lemma 3.6(iv), so W is an (A, A^*)-module. Applying Lemma 3.3, we find $W = 0$ or $W = V$. Observe W contains V_{00}, and $V_{00} = E_0^*V$ is nonzero, so $W \neq 0$. It follows $W = V$, and in other words

$$V = V_{00} + V_{11} + \cdots + V_{dd}. \tag{35}$$

We show the sum (35) is direct. To do this, we show

$$(V_{00} + V_{11} + \cdots + V_{i-1,i-1}) \cap V_{ii} = 0$$

for $1 \leq i \leq d$. Let the integer i be given. By (27) we find

$$V_{jj} \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V$$

for $0 \leq j \leq i-1$, and

$$V_{ii} \subseteq E_iV + E_{i+1}V + \cdots + E_dV.$$

It follows

$$\begin{aligned}
&(V_{00} + V_{11} + \cdots + V_{i-1,i-1}) \cap V_{ii} \\
&\quad \subseteq (E_0^*V + E_1^*V + \cdots + E_{i-1}^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV) \\
&\quad = V_{i-1,i} \\
&\quad = 0
\end{aligned}$$

in view of Lemma 3.7. We have now shown the sum (35) is direct, so (34) holds. It remains to show (33). In view of (34), it suffices to show $V_{ii} \neq 0$ for $0 \leq i \leq d$. Suppose there exists an integer i $(0 \leq i \leq d)$ such that $V_{ii} = 0$. We observe $i \neq 0$, since $V_{00} = E_0^*V$ is nonzero, and $i \neq d$, since $V_{dd} = E_dV$ is nonzero. Set

$$U = V_{00} + V_{11} + \cdots + V_{i-1,i-1},$$

and observe $U \neq 0$ and $U \neq V$ by our remarks above. By Lemma 3.6(ii) and since $V_{ii} = 0$, we find $AU \subseteq U$. By Lemma 3.6(iv) we find $A^*U \subseteq U$. Now U is an (A, A^*)-module. Applying Lemma 3.3, we find $U = 0$ or $U = V$, contradicting our comments above. We conclude $V_{ii} \neq 0$ for $0 \leq i \leq d$, and (33) follows.
□

Lemma 3.9 *With reference to Definition 3.1 and Definition 3.4, the following (i)–(iv) hold.*

(i) $(A - \theta_i I)V_{ii} = V_{i+1,i+1} \qquad (0 \leq i \leq d-1)$,

(ii) $(A - \theta_d I)V_{dd} = 0$,

(iii) $(A^* - \theta_i^* I)V_{ii} = V_{i-1,i-1} \qquad (1 \leq i \leq d)$,

(iv) $(A^* - \theta_0^* I)V_{00} = 0$.

Proof: (i) Let the integer i be given. Recall $(A - \theta_i I)V_{ii}$ is contained in $V_{i+1,i+1}$ by Lemma 3.6(i), and $V_{i+1,i+1}$ has dimension 1 by (33), so it suffices to show

$$(A - \theta_i I)V_{ii} \neq 0. \tag{36}$$

Assume $(A - \theta_i I)V_{ii} = 0$, and set

$$W = V_{00} + V_{11} + \cdots + V_{ii}.$$

By Lemma 3.8, and since $0 \leq i < d$, we find $W \neq 0$ and $W \neq V$. Observe $AV_{ii} \subseteq V_{ii}$ by our above assumption; combining this with Lemma 3.6(ii), we find $AW \subseteq W$. By Lemma 3.6(iv), we find $A^*W \subseteq W$. Now W is an (A, A^*)-module. Applying Lemma 3.3, we find $W = 0$ or $W = V$, contradicting our above remarks. We conclude (36) holds, and the result follows.
(ii) Recall $V_{dd} = E_d V$ by Lemma 3.5(ii).
(iii), (iv) Apply (i),(ii) above to $\Phi^{\downarrow\Downarrow *}$.
$\square$
Proof of Theorem 3.2: We first show there exists an isomorphism of $\mathcal{F}$-algebras $\flat : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathcal{F})$ and there exists nonzero scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathcal{F}$ such that (25) holds. Let V denote the irreducible left $\mathcal{A}$-module, and let the subspaces V_{ij} of V be as in Definition 3.4. Let v_0 denote a nonzero vector in $V_{00} = E_0^* V$, and let $v_1, v_2, \ldots, v_d$ denote the vectors in V satisfying

$$(A - \theta_i I)v_i = v_{i+1} \qquad (0 \leq i \leq d-1). \tag{37}$$

Combining Lemma 3.9(i) and (33), we find v_i is a basis for V_{ii}, for $0 \leq i \leq d$. By this and (34), we find

$$v_0, v_1, \ldots, v_d \tag{38}$$

is a basis for V. For all $X \in \mathcal{A}$, let $X^\flat$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathcal{F})$ that represents the X action on V with respect to the basis (38). By elementary linear algebra, we find the map $\flat : X \to X^\flat$ is an isomorphism of $\mathcal{F}$-algebras from $\mathcal{A}$ to $\mathrm{Mat}_{d+1}(\mathcal{F})$. We check $A^\flat$ and $A^{*\flat}$ have the required form (25). To get $A^\flat$, we consider the action of A on the basis (38). Most of this action is given in (37), but we still need Av_d. Recall v_d is contained in V_{dd}, so by Lemma 3.9(ii),

$$Av_d = \theta_d v_d. \tag{39}$$

Combining (37), (39), we find the equation on the left in (25) holds. To get $A^{*\flat}$, we consider the action of A^* on the basis (38). By Lemma 3.9(iii), there exist nonzero scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ in $\mathcal{F}$ such that

$$(A^* - \theta_i^* I)v_i = \varphi_i v_{i-1} \qquad (1 \leq i \leq d). \tag{40}$$

By Lemma 3.9(iv),

$$(A^* - \theta_0^* I)v_0 = 0. \tag{41}$$

Combining (40), (41), we obtain the equation on the right in (25).
Concerning uniqueness, let $\flat' : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathcal{F})$ denote an isomorphism of $\mathcal{F}$-algebras, and let $\varphi_1', \varphi_2', \ldots, \varphi_d'$ denote scalars in $\mathcal{F}$ such that (25) hold. We show $\flat = \flat'$. To do this, we show the composition $\sigma = \flat^{-1}\flat'$ equals 1. Observe σ is an isomorphism of $\mathcal{F}$-algebras from $\mathrm{Mat}_{d+1}(\mathcal{F})$ to itself, so by elementary linear algebra, there exists an invertible matrix X in $\mathrm{Mat}_{d+1}(\mathcal{F})$ such that

$$Y^\sigma = X^{-1}YX \qquad (\forall Y \in \mathrm{Mat}_{d+1}(\mathcal{F})). \tag{42}$$

Apparently

$$XY^\sigma = YX \qquad (\forall Y \in \mathrm{Mat}_{d+1}(\mathcal{F})). \tag{43}$$

If $Y = A^{*\flat}$ then $Y^\sigma = A^{*\flat'}$; using these values in (43), we get

$$XA^{*\flat'} = A^{*\flat}X. \tag{44}$$

Both $A^{*\flat}$ and $A^{*\flat'}$ are upper triangular, and each has ii entry θ^*_i for $0 \leq i \leq d$; multiplying out each side of (44) using this, and recalling $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are distinct, we find X is upper triangular. If $Y = A^\flat$ then $Y^\sigma = A^{\flat'}$; using these values in (43), we get

$$XA^{\flat'} = A^\flat X. \tag{45}$$

The matrices $A^\flat$ and $A^{\flat'}$ are equal, and are given by the equation on the left in (25); multiplying out each side of (45) using this, and recalling $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, we find X is a scalar multiple of the identity. It follows $\sigma = 1$, so $\flat = \flat'$. The scalars φ'_i are determined by Φ and $\flat'$, so $\varphi'_i = \varphi_i$ for $1 \leq i \leq d$. This completes the proof.
□

Definition 3.10 *With reference to Definition 3.1, by the split canonical form of Φ, we mean the Leonard system $\Phi^\flat$, where $\flat = \flat(\Phi)$ is from Theorem 3.2. By the φ-sequence of Φ, we mean the seqence $\varphi_1, \varphi_2, \ldots, \varphi_d$ from Theorem 3.2. For notational convenience, we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$.*

Lemma 3.11 *Let Φ and Φ' denote a Leonard systems over $\mathcal{F}$. Then the following are equivalent.*

(i) Φ and Φ' are isomorphic.

(ii) Φ and Φ' share the same eigenvalue sequence, dual eigenvalue sequence, and φ-sequence.

Proof: *(i)* $\rightarrow$ *(ii)* Routine.
(ii) $\rightarrow$ *(i)* Observe Φ and Φ' are both isomorphic to a common split canonical form, so they are isomorphic.
□
It is helpful to consider the following parameters.

Definition 3.12 *With reference to Definition 3.1, let $\phi_1, \phi_2, \ldots, \phi_d$ denote the φ-sequence of $\Phi^\Downarrow$. Let the map $\flat = \flat(\Phi)$ be as in Theorem 3.2, and let $\diamondsuit = \flat^\Downarrow$ denote the corresponding map for $\Phi^\Downarrow$. Observe*

$$A^\diamondsuit = \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \qquad A^{*\diamondsuit} = \begin{pmatrix} \theta^*_0 & \phi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \phi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

We call $\phi_1, \phi_2, \ldots, \phi_d$ the ϕ-sequence of Φ. For notational convenience, we define $\phi_0 = 0$, $\phi_{d+1} = 0$.

Lemma 3.13 *Let Φ and Φ' denote a Leonard systems over $\mathcal{F}$. Then the following are equivalent.*

(i) Φ and Φ' are isomorphic.

(ii) Φ and Φ' share the same eigenvalue sequence, dual eigenvalue sequence, and ϕ-sequence.

Proof: Apply Lemma 3.11 to the second inversions of Φ and Φ'.
□

4 The primitive idempotents of a Leonard system

In this section, we consider a Leonard system in split canonical form, and compute the entries of the primitive idempotents. We begin by considering a more general situation.

Definition 4.1 *Let d denote a nonnegative integer and let $\mathcal{F}$ denote a field. In this section, we let A and A^* denote any matrices in $Mat_{d+1}(\mathcal{F})$ of the form*

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix},$$

such that

$$\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \qquad \text{if} \quad i \neq j \qquad (0 \leq i, j \leq d), \tag{46}$$

$$\varphi_i \neq 0 \qquad (1 \leq i \leq d). \tag{47}$$

We observe A (resp. A^*) is multiplicity-free, with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$). For $0 \leq i \leq d$, we let E_i (resp. E^*_i) denote the primitive idempotent for A (resp. A^*) associated with θ_i (resp. θ^*_i). For notational convenience, we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$.

With reference to Definition 4.1, we do not assume A, A^* come from a Leonard system, so we do not expect any D_4 action; however we do have the following result.

Lemma 4.2 *With reference to Definition 4.1, let G denote the diagonal matrix in $Mat_{d+1}(\mathcal{F})$ with diagonal entries*

$$G_{ii} = \varphi_1 \varphi_2 \cdots \varphi_i \qquad (0 \leq i \leq d). \tag{48}$$

Let Z denote the matrix in $Mat_{d+1}(\mathcal{F})$ with ij^{th} entry 1 if $i + j = d$, and 0 if $i + j \neq d$, for $0 \leq i, j \leq d$. Then (i)–(iii) hold below.

(i) The matrices $G^{-1}A^{*t}G$ and $G^{-1}A^tG$ equal

$$\begin{pmatrix} \theta^*_0 & & & & & \mathbf{0} \\ 1 & \theta^*_1 & & & & \\ & 1 & \theta^*_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta^*_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta_1 & \varphi_2 & & & \\ & & \theta_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d \end{pmatrix},$$

respectively.

(ii) The matrices ZA^tZ and $ZA^{*t}Z$ equal

$$\begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \theta^*_d & \varphi_d & & & & \mathbf{0} \\ & \theta^*_{d-1} & \varphi_{d-1} & & & \\ & & \theta^*_{d-2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_1 \\ \mathbf{0} & & & & & \theta^*_0 \end{pmatrix},$$

respectively.

(iii) The matrices $ZGA^*G^{-1}Z$ and $ZGAG^{-1}Z$ equal

$$\left(\begin{array}{cccccc} \theta^*_d & & & & & \mathbf{0} \\ 1 & \theta^*_{d-1} & & & & \\ & 1 & \theta^*_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta^*_0 \end{array}\right), \qquad \left(\begin{array}{cccccc} \theta_d & \varphi_d & & & & \mathbf{0} \\ & \theta_{d-1} & \varphi_{d-1} & & & \\ & & \theta_{d-2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_1 \\ \mathbf{0} & & & & & \theta_0 \end{array}\right),$$

respectively.

We remark $Z = Z^{-1}$.

Proof: Routine matrix multiplication.
□
We remark that in the above lemma only (i) and (iii) will be used later in the paper; we include (ii) for the sake of completeness.
It is convenient to use the following notation.

Definition 4.3 *Suppose we are given a nonnegative integer d and a sequence of scalars*

$$\theta_0, \theta_1, \ldots, \theta_d; \qquad\qquad \theta^*_0, \theta^*_1, \ldots, \theta^*_d$$

taken from a field $\mathcal{F}$. Then for $0 \leq i \leq d+1$, we let τ_i, τ^*_i, η_i, η^*_i denote the following polynomials in $\mathcal{F}[\lambda]$.

$$\begin{aligned} \tau_i &:= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}), && (49)\\ \tau^*_i &:= (\lambda - \theta^*_0)(\lambda - \theta^*_1)\cdots(\lambda - \theta^*_{i-1}), && (50)\\ \eta_i &:= (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}), && (51)\\ \eta^*_i &:= (\lambda - \theta^*_d)(\lambda - \theta^*_{d-1})\cdots(\lambda - \theta^*_{d-i+1}). && (52) \end{aligned}$$

We observe each of τ_i, τ^*_i, η_i, η^*_i is monic of degree i.

Lemma 4.4 *With reference to Definition 4.1, Pick any integer r $(0 \leq r \leq d)$. Then the primitive idempotent E_r of A has ij^{th} entry*

$$\frac{\tau_j(\theta_r)\eta_{d-i}(\theta_r)}{\tau_r(\theta_r)\eta_{d-r}(\theta_r)} \qquad (53)$$

for $0 \leq i, j \leq d$. We are using the notation (49), (51).

Proof: Let the integers i, j be given. Computing the ij entry of $AE_r = \theta_r E_r$ using matrix multiplication, and taking into account the form of A in Definition 4.1, we find

$$(E_r)_{i-1,j} = (\theta_r - \theta_i)(E_r)_{ij}$$

if $i \geq 1$. Replacing i by $i+1$ in the above line, we find

$$(E_r)_{ij} = (\theta_r - \theta_{i+1})(E_r)_{i+1,j} \qquad (54)$$

if $i \leq d-1$. Using the recursion (54), we routinely find

$$\begin{aligned} (E_r)_{ij} &= (\theta_r - \theta_{i+1})(\theta_r - \theta_{i+2})\cdots(\theta_r - \theta_d)(E_r)_{dj} \\ &= \eta_{d-i}(\theta_r)(E_r)_{dj}. && (55) \end{aligned}$$

Computing the dj entry of $E_rA = \theta_r E_r$ using matrix multiplication, and taking into account the form of A, we find

$$(E_r)_{d,j+1} = (\theta_r - \theta_j)(E_r)_{dj}$$

if $j \leq d-1$. Replacing j by $j-1$ in the above line we find

$$(E_r)_{dj} = (\theta_r - \theta_{j-1})(E_r)_{d,j-1} \tag{56}$$

if $j \geq 1$. Using the recursion (56), we routinely find

$$\begin{aligned} (E_r)_{dj} &= (\theta_r - \theta_{j-1})(\theta_r - \theta_{j-2}) \cdots (\theta_r - \theta_0)(E_r)_{d0} \\ &= \tau_j(\theta_r)(E_r)_{d0}. \end{aligned} \tag{57}$$

Combining (55), (57), we find

$$(E_r)_{ij} = \tau_j(\theta_r)\eta_{d-i}(\theta_r)c, \tag{58}$$

where we abbreviate $c = (E_r)_{d0}$. We now find c. Since A is lower triangular, and since E_r is a polynomial in A, we see E_r is lower triangular. Recall $E_r^2 = E_r$, so the diagonal entry of $(E_r)_{rr}$ equals 0 or 1. We show $(E_r)_{rr} = 1$. Setting $i = r$, $j = r$ in (58),

$$(E_r)_{rr} = \tau_r(\theta_r)\eta_{d-r}(\theta_r)c. \tag{59}$$

Observe $\tau_r(\theta_r) \neq 0$ and $\eta_{d-r}(\theta_r) \neq 0$ by Definition 4.3, and since the eigenvalues are distinct. Observe $c \neq 0$; otherwise $E_r = 0$ in view of (58). Apparently the right side of (59) is not 0, so $(E_r)_{rr} \neq 0$, and we conclude $(E_r)_{rr} = 1$. Setting $(E_r)_{rr} = 1$ in (59), solving for c, and evaluating (58) using the result, we find the ij^{th} entry of E_r is given by (53).
$\square$

Example 4.5 *With reference to Definition 4.1, for $d = 2$, the primitive idempotents of A are as follows.*

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\theta_0-\theta_1} & 0 & 0 \\ \frac{1}{(\theta_0-\theta_1)(\theta_0-\theta_2)} & 0 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\theta_1-\theta_0} & 1 & 0 \\ \frac{1}{(\theta_1-\theta_0)(\theta_1-\theta_2)} & \frac{1}{\theta_1-\theta_2} & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{(\theta_2-\theta_1)(\theta_2-\theta_0)} & \frac{1}{\theta_2-\theta_1} & 1 \end{pmatrix}.$$

Lemma 4.6 *With reference to Definition 4.1, Pick any integer r $(0 \leq r \leq d)$. Then the primitive idempotent E_r^* of A^* has ij^{th} entry*

$$\frac{\varphi_1\varphi_2\cdots\varphi_j}{\varphi_1\varphi_2\cdots\varphi_i}\,\frac{\tau_i^*(\theta_r^*)\eta_{d-j}^*(\theta_r^*)}{\tau_r^*(\theta_r^*)\eta_{d-r}^*(\theta_r^*)} \tag{60}$$

for $0 \leq i, j \leq d$. We are using the notation (50), (52).

Proof: Let the matrix G be as in Lemma 4.2, and set $A' := G^{-1}A^{*t}G$; this matrix is given on the left in Lemma 4.2(i). Let E'_r denote the primitive idempotent of A' associated with the eigenvalue θ^*_r. We find E'_r in two ways. On one hand, applying Lemma 4.4 to A', we find E'_r has ij^{th} entry

$$\frac{\tau^*_j(\theta^*_r)\eta^*_{d-i}(\theta^*_r)}{\tau^*_r(\theta^*_r)\eta^*_{d-r}(\theta^*_r)} \tag{61}$$

for $0 \leq i, j \leq d$. On the other hand, by elementary linear algebra

$$E'_r = G^{-1}E^{*t}_r G,$$

so E'_r has ij^{th} entry

$$G^{-1}_{ii}(E^*_r)_{ji}G_{jj} = \frac{\varphi_1\varphi_2\cdots\varphi_j}{\varphi_1\varphi_2\cdots\varphi_i}(E^*_r)_{ji} \tag{62}$$

for $0 \leq i, j \leq d$. Equating (61) and the right side of (62), and solving for $(E^*_r)_{ji}$, we routinely obtain the result.
□

Example 4.7 *With reference to Definition 4.1, for $d = 2$, the primitive idempotents of A^* are given by*

$$E^*_0 = \begin{pmatrix} 1 & \frac{\varphi_1}{\theta^*_0-\theta^*_1} & \frac{\varphi_1\varphi_2}{(\theta^*_0-\theta^*_1)(\theta^*_0-\theta^*_2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E^*_1 = \begin{pmatrix} 0 & \frac{\varphi_1}{\theta^*_1-\theta^*_0} & \frac{\varphi_1\varphi_2}{(\theta^*_1-\theta^*_0)(\theta^*_1-\theta^*_2)} \\ 0 & 1 & \frac{\varphi_2}{\theta^*_1-\theta^*_2} \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^*_2 = \begin{pmatrix} 0 & 0 & \frac{\varphi_1\varphi_2}{(\theta^*_2-\theta^*_1)(\theta^*_2-\theta^*_0)} \\ 0 & 0 & \frac{\varphi_2}{\theta^*_2-\theta^*_1} \\ 0 & 0 & 1 \end{pmatrix}.$$

We now restate Lemma 4.4 and Lemma 4.6 in the context of of Leonard systems.

Theorem 4.8 *Let $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$ denote a Leonard system, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, and φ-sequence $\varphi_1, \varphi_2, \ldots \varphi_d$. Let the map $\flat = \flat(\Phi)$ be as in Theorem 3.2. Pick any integer r $(0 \leq r \leq d)$. Then $E^\flat_r$ is the matrix in $Mat_{d+1}(\mathcal{F})$ with ij^{th} entry*

$$\frac{\tau_j(\theta_r)\eta_{d-i}(\theta_r)}{\tau_r(\theta_r)\eta_{d-r}(\theta_r)} \tag{63}$$

for $0 \leq i, j \leq d$. Moreover, $E^{*\flat}_r$ is the matrix in $Mat_{d+1}(\mathcal{F})$ with ij^{th} entry

$$\frac{\varphi_1\varphi_2\cdots\varphi_j}{\varphi_1\varphi_2\cdots\varphi_i}\frac{\tau^*_i(\theta^*_r)\eta^*_{d-j}(\theta^*_r)}{\tau^*_r(\theta^*_r)\eta^*_{d-r}(\theta^*_r)} \tag{64}$$

for $0 \leq i, j \leq d$. We are using the notation (49)–(52).

Proof: By Theorem 3.2, the matrices $A^\flat$ and $A^{*\flat}$ are of the form given in Definition 4.1. To get $E^\flat_r$, apply Lemma 4.4 to $A^\flat$, and observe $E^\flat_r$ is the primitive idempotent $A^\flat$ associated with θ_r. To get $E^{*\flat}_r$, apply Lemma 4.6 to $A^{*\flat}$, and observe $E^{*\flat}_r$ is the primitive idempotent $A^{*\flat}$ associated with θ^*_r.
□

We finish this section with a few observations.

Lemma 4.9 *With reference to Definition 4.1,*

(i) $E_i A^* E_j = \begin{cases} 0 & \text{if } j - i > 1; \\ \neq 0 & \text{if } j - i = 1 \end{cases} \qquad (0 \leq i, j \leq d),$

(ii) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } i - j > 1; \\ \neq 0 & \text{if } i - j = 1 \end{cases} \qquad (0 \leq i, j \leq d).$

Proof: (i) For $0 \leq r \leq d$, consider the pattern of zero entries in E_r. By Lemma 4.4, we find E_r has all entries 0 in rows $0, 1, \ldots, r-1$ and columns $r+1, r+2, \ldots, d$. Moreover, the rr entry of E_r is 1. Multiplying out $E_i A^* E_j$ using this and the shape of A^*, we find that if $j - i > 1$, then all entries are 0, and if $j - i = 1$, then the ij entry is φ_j and hence nonzero. The result follows.
(ii) Very similar to the proof of (i) above.
□

Lemma 4.10 *With reference to Definition 4.1, the following are equivalent.*

(i) (A, A^*) *is a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathcal{F})$.

(ii) $(A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$ *is a Leonard system in* $\mathrm{Mat}_{d+1}(\mathcal{F})$.

Suppose (i), (ii) hold. Then the Leonard system in (ii) has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, *dual eigenvalue sequence* $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, *and* φ*-sequence* $\varphi_1, \varphi_2, \ldots, \varphi_d$.

Proof: Let Φ denote the sequence in part (ii).
$(i) \rightarrow (ii)$ We verify Φ satisfies the conditions (i)–(v) of Definition 1.4, where we take $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathcal{F})$. Conditions (i)–(iii) are immediate from the construction, so consider conditions (iv), (v). We assume (A, A^*) is a Leonard pair, so it is associated with some Leonard system. It follows

$$E_i A^* E_j = 0 \quad \text{iff} \quad E_j A^* E_i = 0 \qquad (0 \leq i, j \leq d), \tag{65}$$

$$E_i^* A E_j^* = 0 \quad \text{iff} \quad E_j^* A E_i^* = 0 \qquad (0 \leq i, j \leq d). \tag{66}$$

Combining (65), (66) with Lemma 4.9, we find Φ satisfies the conditions (iv), (v) of Definition 1.4. We have now shown Φ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathcal{F})$.
$(ii) \rightarrow (i)$ Immediate from Definition 1.6.
Now suppose (i), (ii) hold. From the construction Φ has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. From the form of the matrices A and A^* in Definition 4.1, we find the map $\flat = \flat(\Phi)$ from Theorem 3.2 is the identity. Now Φ has φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ in view of Definition 3.10.
□

5 A formula for the φ_i

In this section, we continue to consider the situation of Definition 4.1. We obtain the scalars φ_i from that definition in terms of the scalars a_i, a_i^* introduced in Definition 2.5. To do this, we first obtain the a_i and a_i^* in terms of the φ_i.

Lemma 5.1 *With reference to Definition 4.1 and Definition 2.5,*

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \qquad (0 \leq i \leq d), \tag{67}$$

where we recall $\varphi_0 = 0$, $\varphi_{d+1} = 0$, *and where* θ^*_{-1}, θ^*_{d+1} *denote indeterminants. Moreover,*

$$a^*_i = \theta^*_i + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \qquad (0 \le i \le d), \tag{68}$$

where θ_{-1}, θ_{d+1} *denote indeterminants.*

Proof: Concerning (67), recall a_i equals the trace of AE^*_i, and this is the sum of the diagonal entries in AE^*_i. Computing these entries by matrix multiplication, and taking into account the form of A in Definition 4.1, we find that for $0 \le j \le d$, the jj entry

$$(AE^*_i)_{jj} \quad = \quad \theta_j (E^*_i)_{jj} + (E^*_i)_{j-1,j} \tag{69}$$

where we interpret the term on the right in (69) to be zero if $j = 0$. By Lemma 4.6, the diagonal entry $(E^*_i)_{jj}$ equals 1 if $j = i$, and 0 if $j \neq i$. Moreover, the entry $(E^*_i)_{j-1,j}$ equals $\varphi_i(\theta^*_i - \theta^*_{i-1})^{-1}$ if $j = i$, $\varphi_{i+1}(\theta^*_i - \theta^*_{i+1})^{-1}$ if $j = i+1$, and 0 if $j \notin \{i, i+1\}$. Evaluating (69) using the above information, we readily obtain (67). The proof of (68) is very similar, and omitted.
□

Lemma 5.2 *With reference to Definition 4.1 and Definition 2.5, pick any integer* i $(1 \le i \le d)$*. Then the scalar* φ_i *equals each of the following four expressions.*

$$(\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_h - a_h), \qquad (\theta^*_{i-1} - \theta^*_i) \sum_{h=i}^{d} (\theta_h - a_h), \tag{70}$$

$$(\theta_i - \theta_{i-1}) \sum_{h=0}^{i-1} (\theta^*_h - a^*_h), \qquad (\theta_{i-1} - \theta_i) \sum_{h=i}^{d} (\theta^*_h - a^*_h). \tag{71}$$

Proof: To see φ_i equals the expression on the left in (70), eliminate each of $a_0, a_1, \ldots, a_{i-1}$ in that expression using (67), and simplify. The two expressions in (70) are equal by (23). To see φ_i equals the expression on the left in (71), eliminate each of $a^*_0, a^*_1, \ldots, a^*_{i-1}$ in that expression using (68), and simplify. The two expressions in (71) are equal by (24).
□

6 The D_4 action

Definition 6.1 *In this section, d will denote a nonnegative integer, $\mathcal{F}$ will denote a field, and $\mathcal{A}$ will denote an $\mathcal{F}$-algebra isomorphic to* $Mat_{d+1}(\mathcal{F})$*. We let*

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d) \tag{72}$$

denote a Leonard system in $\mathcal{A}$, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$.

With reference to Definition 6.1, we now consider what happens to the $\theta_i, \theta^*_i, \varphi_i, \phi_i$ when Φ is replaced by a relative. We begin with a simple observation.

Lemma 6.2 *With reference to Definition 6.1 and Definition 2.5,*

$$\theta^{\downarrow}_i = \theta_i, \qquad \theta^{\Downarrow}_i = \theta_{d-i}, \tag{73}$$

$$a^{\downarrow}_i = a_{d-i}, \qquad a^{\Downarrow}_i = a_i, \tag{74}$$

for $0 \le i \le d$. (We are using the notation of Definition 1.10).

Proof: Recall θ_i is the eigenvalue of A associated with E_i, and a_i is the trace of AE^*_i. By Definition 1.10, we find that for all $g \in D_4$, θ^g_i is the eigenvalue of A^g associated with E^g_i, and a^g_i is the trace of $A^g E^{*g}_i$. By (9) we have $A^{\downarrow} = A$, $E^{\downarrow}_i = E_i$, and $E^{*\downarrow}_i = E^*_{d-i}$. By (10) we have $A^{\Downarrow} = A$, $E^{\Downarrow}_i = E_{d-i}$, and $E^{*\Downarrow}_i = E^*_i$. The result follows.
□

Lemma 6.3 *With reference to Definition 6.1 and Definition 2.5, for* $1 \leq i \leq d$*, the scalar* φ_i *equals each of the following four expressions.*

$$(\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_h - a_h), \qquad (\theta^*_{i-1} - \theta^*_i) \sum_{h=i}^{d} (\theta_h - a_h), \tag{75}$$

$$(\theta_i - \theta_{i-1}) \sum_{h=0}^{i-1} (\theta^*_h - a^*_h), \qquad (\theta_{i-1} - \theta_i) \sum_{h=i}^{d} (\theta^*_h - a^*_h). \tag{76}$$

Proof: Apply Lemma 5.2 to the split canonical form of Φ.
□

Lemma 6.4 *With reference to Definition 6.1 and Definition 2.5, for* $1 \leq i \leq d$*, the scalar* ϕ_i *equals each of the following four expressions.*

$$(\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_{d-h} - a_h), \qquad (\theta^*_{i-1} - \theta^*_i) \sum_{h=i}^{d} (\theta_{d-h} - a_h), \tag{77}$$

$$(\theta_{d-i} - \theta_{d-i+1}) \sum_{h=0}^{i-1} (\theta^*_h - a^*_{d-h}), \qquad (\theta_{d-i+1} - \theta_{d-i}) \sum_{h=i}^{d} (\theta^*_h - a^*_{d-h}). \tag{78}$$

Proof: By definition $\phi_i = \varphi^{\Downarrow}_i$. To find $\varphi^{\Downarrow}_i$, apply Lemma 6.3 to $\Phi^{\Downarrow}$, and evaluate the result using (11), (12), and Lemma 6.2.
□

We are now ready to prove Theorem 1.11 from the Introduction.

Proof of Theorem 1.11: Referring to the table in the theorem statement, the θ^g_i and θ^{*g}_i are readily obtained using (11), (12), and (73). Comparing the expressions on the left in (75), (76), we see $\varphi^*_i = \varphi_i$. By definition $\varphi^{\Downarrow}_i = \phi_i$, and recall $\Downarrow$ is an involution, so $\phi^{\Downarrow}_i = \varphi_i$. Applying $*$ to the expression on the left in (77), and replacing i by $d-i+1$ in the result, we get the expression on the right in (78); it follows $\phi^*_i = \phi_{d-i+1}$. Applying $\downarrow$ to the expression on the left in (75), and replacing i by $d-i+1$ in the result, we get the expression on the right in (77); it follows $\varphi^{\downarrow}_i = \phi_{d-i+1}$. By this and since $\downarrow$ is an involution, we find $\phi^{\downarrow}_i = \varphi_{d-i+1}$. The remaining entries of the table are routinely obtained using (11), (12).
□

We interpret the data in Theorem 1.11 as follows. Let Φ be as in Definition 6.1. By the *parameter square* of Φ, we mean the diagram below.

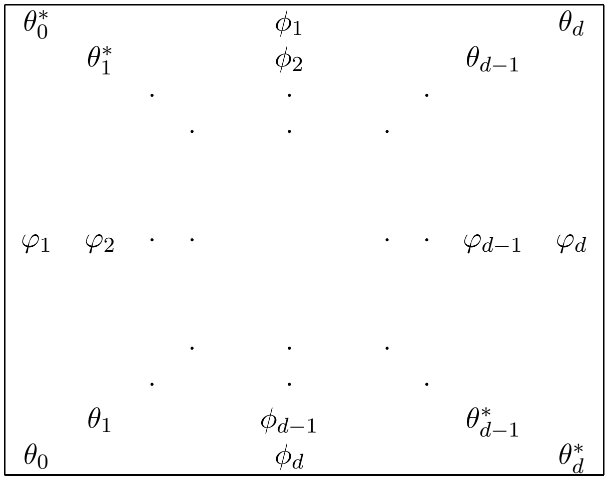

Fig. 1. The parameter square of a Leonard system

To get the parameter square for Φ^*, reflect the parameter square of Φ about the horizontal line running through the center. To get the parameter square of $\Phi^{\downarrow}$, reflect the parameter square of Φ about the diagonal running from the bottom left corner to the top right corner. To get the parameter square of $\Phi^{\Downarrow}$, reflect the parameter square of Φ about the diagonal running from the bottom right corner to the top left corner.

We finish this section with a comment.

Lemma 6.5 *With reference to Definition 6.1, for $1 \leq i \leq d$,*

(i) $\varphi_i - \phi_i \;=\; (\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_h - \theta_{d-h}),$

(ii) $\varphi_i - \phi_{d-i+1} \;=\; (\theta_i - \theta_{i-1}) \sum_{h=0}^{i-1} (\theta^*_h - \theta^*_{d-h}).$

Proof:(i) Subtract the expression on the left in (77) from the expression on the left in (75).
(ii) Apply (i) above to Φ^*.
□

7 A result on reducibility

In this section, we return to the situation of Definition 4.1. We extend the domain of definition of the ϕ_i scalars to the level of Definition 4.1 using Lemma 6.5(i). We use the resulting constants to get a reducibility result.

Definition 7.1 *With reference to Definition 4.1, we define*

$$\phi_i = \varphi_i - (\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_h - \theta_{d-h}) \qquad (1 \leq i \leq d). \tag{79}$$

For notational convenience, we set $\phi_0 = 0$, $\phi_{d+1} = 0$.

Lemma 7.2 *With reference to Definition 4.1, let $V = \mathcal{F}^{d+1}$ denote the irreducible left module for $Mat_{d+1}(\mathcal{F})$, and let W denote a nonzero (A, A^*)-module in V. Then there exists an integer r $(0 \le r \le d)$ such that both*

$$W = \sum_{h=r}^{d} E_h^* V, \qquad W = \sum_{h=0}^{d-r} E_h V. \tag{80}$$

Moreover, the scalar ϕ_r from Definition 7.1 is zero.

Proof: Since W is nonzero and $A^*W \subseteq W$, there exists a nonempty subset S^* of $\{0, 1, \ldots, d\}$ such that $W = \sum_{i \in S^*} E_i^* V$. Recall by Lemma 4.9(ii) that $E_{i+1}^* A E_i^* \ne 0$ for $0 \le i \le d-1$. Combining this with Lemma 2.4(ii), we find $i \in S^*$ implies $i+1 \in S^*$ for $0 \le i \le d-1$. It follows $S^* = \{r, r+1, \ldots, d\}$ for some integer r $(0 \le r \le d)$. Since W is nonzero and $AW \subseteq W$, there exists a nonempty subset S of $\{0, 1, \ldots, d\}$ such that $W = \sum_{i \in S} E_i V$. Recall by Lemma 4.9(i) that $E_{i-1} A^* E_i \ne 0$ for $1 \le i \le d$. Combining this with Lemma 2.3(ii), we find $i \in S$ implies $i-1 \in S$ for $1 \le i \le d$. It follows $S = \{0, 1, \ldots, s\}$ for some integer s $(0 \le s \le d)$. Considering the dimension of W we find $|S| = |S^*|$, so $s = d - r$, and (80) follows. It remains to show $\phi_r = 0$. This holds by definition if $r = 0$, so assume $r \ge 1$. To get $\phi_r = 0$ in this case, we first show

$$a_r + a_{r+1} + \cdots + a_d = \theta_0 + \theta_1 + \cdots + \theta_{d-r}. \tag{81}$$

For convenience, we abbreviate $E = \sum_{h=0}^{d-r} E_h$ and $E^* = \sum_{h=r}^{d} E_h^*$. We show AE and AE^* have the same trace. To do this, we put $X = A(E - E^*)$, and show X has trace 0. In fact $X^2 = 0$. To see this, we show $XV \subseteq W$ and $XW = 0$. Each of EV, E^*V equals W by (80), so $(E - E^*)V \subseteq W$. Recall $AW \subseteq W$, so $XV \subseteq W$. Observe each of E, E^* acts as the identity on W, so $(E - E^*)W = 0$, and it follows $XW = 0$. We have now shown $X^2 = 0$, so X has trace 0, and AE, AE^* have the same trace. We now compute these traces. By (2) and since each E_h has trace 1, we find AE has trace $\sum_{h=0}^{d-r} \theta_h$. Using Definition 2.5, we routinely find AE^* has trace $\sum_{h=r}^{d} a_h$. We now have (81). Eliminating the left side of (81) using the equation on the right in (70), we find $\phi_r = 0$.
□

Theorem 7.3 *With reference to Definition 4.1, let $V = \mathcal{F}^{d+1}$ denote the irreducible left module for $Mat_{d+1}(\mathcal{F})$, and suppose the scalars $\phi_1, \phi_2, \ldots, \phi_d$ from (79) are all nonzero. Then V is irreducible as an (A, A^*)-module.*

Proof: Let W denote a nonzero (A, A^*)-module in V. We show $W = V$. Let r denote the integer associated with W from Lemma 7.2. From that lemma and our present assumption we find r is not one of $1, 2, \ldots, d$, so $r = 0$. Setting $r = 0$ in (80), we find $W = V$.
□

8 Recurrent sequences

It is going to turn out that the eigenvalue sequence and dual eigenvalue sequence of a Leonard system each satisfy a certain recurrence. In this section, we set the stage by considering this recurrence from several points of view.

Definition 8.1 *In this section, $\mathcal{F}$ will denote a field, d will denote a nonnegative integer, and $\theta_0, \theta_1, \ldots, \theta_d$ will denote a sequence of scalars taken from $\mathcal{F}$.*

Definition 8.2 *With reference to Definition 8.1, let β, γ, ϱ denote scalars in $\mathcal{F}$.*

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be recurrent whenever $\theta_{i-1} \neq \theta_i$ for $2 \leq i \leq d-1$, and

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \tag{82}$$

is independent of i, for $2 \leq i \leq d-1$.

(ii) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be β-recurrent whenever

$$\theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{i+1} \tag{83}$$

is zero for $2 \leq i \leq d-1$.

(iii) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be (β, γ)-recurrent whenever

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \tag{84}$$

for $1 \leq i \leq d-1$.

(iv) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be (β, γ, ϱ)-recurrent whenever

$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) = \varrho \tag{85}$$

for $1 \leq i \leq d$.

Lemma 8.3 *With reference to Definition 8.1, the following are equivalent.*

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent.

(ii) There exists $\beta \in \mathcal{F}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent, and $\theta_{i-1} \neq \theta_i$ for $2 \leq i \leq d-1$.

Suppose (i), (ii), and that $d \geq 3$. Then the common value of (82) equals $\beta + 1$.

Proof: Routine.
□

Lemma 8.4 *With reference to Definition 8.1, the following are equivalent for all $\beta \in \mathcal{F}$.*

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent.

(ii) There exists $\gamma \in \mathcal{F}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is (β, γ)-recurrent.

Proof: *(i)* $\rightarrow$ *(ii)* For $2 \leq i \leq d-1$, the expression (83) is zero by assumption, so

$$\theta_{i-2} - \beta\theta_{i-1} + \theta_i = \theta_{i-1} - \beta\theta_i + \theta_{i+1}.$$

Apparently the left side of (84) is independent of i, and the result follows.
(ii) $\rightarrow$ *(i)* Subtracting the equation (84) at i from the corresponding equation obtained by replacing i by $i-1$, we find (83) is zero for $2 \leq i \leq d-1$.
□

Lemma 8.5 *With reference to Definition 8.1, the following (i),(ii) hold for all $\beta, \gamma \in \mathcal{F}$.*

(i) Suppose $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ)*-recurrent. Then there exists* $\varrho \in \mathcal{F}$ *such that* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ, ϱ)*-recurrent.*

(ii) Suppose $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ, ϱ)*-recurrent, and that* $\theta_{i-1} \neq \theta_{i+1}$ *for* $1 \leq i \leq d-1$*. Then* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ)*-recurrent.*

Proof: Let p_i denote the expression on the left in (85), and observe

$$p_i - p_{i+1} \quad = \quad (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta\theta_i + \theta_{i+1} - \gamma)$$

for $1 \leq i \leq d-1$. Assertions (i), (ii) are both routine consequences of this.
$\square$

9 Recurrent sequences in closed form

In this section, we obtain some formula involving recurrent sequences.

Definition 9.1 *In this section,* $\mathcal{F}$ *will denote a field,* d *will denote a nonnegative integer, and* $\beta, \theta_0, \theta_1, \ldots, \theta_d$ *will denote scalars in* $\mathcal{F}$ *such that* $\theta_0, \theta_1, \ldots, \theta_d$ *is* β*-recurrent. We let* $\mathcal{F}^{cl}$ *denote the algebraic closure of* $\mathcal{F}$*. For all* $q \in \mathcal{F}^{cl}$*, we let* $\mathcal{F}[q]$ *denote the field extention of* $\mathcal{F}$ *generated by* q*.*

Lemma 9.2 *With reference to Definition 9.1, the following (i)–(iv) hold.*

(i) Suppose $\beta \neq 2$*,* $\beta \neq -2$*, and pick* $q \in \mathcal{F}^{cl}$ *such that* $q + q^{-1} = \beta$*. Then there exists scalars* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathcal{F}[q]$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \qquad (0 \leq i \leq d). \tag{86}$$

(ii) Suppose $\beta = 2$ *and char*$(\mathcal{F}) \neq 2$*. Then there exists* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathcal{F}$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 i^2 \qquad (0 \leq i \leq d). \tag{87}$$

(iii) Suppose $\beta = -2$ *and char*$(\mathcal{F}) \neq 2$*. Then there exists* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathcal{F}$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 (-1)^i + \alpha_3 i (-1)^i \qquad (0 \leq i \leq d). \tag{88}$$

(iv) Suppose $\beta = 0$ *and char*$(\mathcal{F}) = 2$*. Then there exists* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathcal{F}$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 \binom{i}{2} \qquad (0 \leq i \leq d), \tag{89}$$

where we interpret the binomial coefficient as follows:

$$\binom{i}{2} = \begin{cases} 0 & \text{if } \ i = 0 \ \text{ or } \ i = 1 \ \ (\text{mod } 4), \\ 1 & \text{if } \ i = 2 \ \text{ or } \ i = 3 \ \ (\text{mod } 4). \end{cases}$$

Proof: (i). We assume $d \geq 2$; otherwise the result is trivial. Let q be given, and consider the equations (86) for $i = 0, 1, 2$. These equations are linear in $\alpha_0, \alpha_1, \alpha_2$. We routinely find the coefficient matrix is nonsingular, so there exist $\alpha_0, \alpha_1, \alpha_2$ in $\mathcal{F}[q]$ such that (86) holds for $i = 0, 1, 2$. Using these scalars, let ε_i denote the left side of (86) minus the right side of (86), for $0 \leq i \leq d$. On one hand, ε_0, ε_1, ε_2 are zero from the construction. On the other hand, one readily checks

$$\varepsilon_{i-2} \;-\; (\beta+1)\varepsilon_{i-1} \;+\; (\beta+1)\varepsilon_i \;-\; \varepsilon_{i+1} = 0$$

for $2 \leq i \leq d-1$. Combining these facts, we find $\varepsilon_i = 0$ for $0 \leq i \leq d$, and the result follows.
(ii)–(iv) Similar to the proof of (i) above.
$\square$

Lemma 9.3 *With reference to Definition 9.1, assume* $\theta_0, \theta_1, \ldots, \theta_d$ *are distinct. Then (i)–(iv) hold below.*

(i) Suppose $\beta \neq 2$, $\beta \neq -2$, *and pick* $q \in \mathcal{F}^{cl}$ *such that* $q + q^{-1} = \beta$. *Then* $q^i \neq 1$ *for* $1 \leq i \leq d$.

(ii) Suppose $\beta = 2$ *and char*$(\mathcal{F}) \neq 2$. *Then char*$(\mathcal{F}) = 0$ *or char*$(\mathcal{F}) > d$.

(iii) Suppose $\beta = -2$ *and char*$(\mathcal{F}) \neq 2$. *Then char*$(\mathcal{F}) = 0$ *or char*$(\mathcal{F}) > d/2$.

(iv) Suppose $\beta = 0$ *and char*$(\mathcal{F}) = 2$. *Then* $d \leq 3$.

Proof: (i) Using (86), we find $q^i = 1$ implies $\theta_i = \theta_0$ for $1 \leq i \leq d$.
(ii) Using (87), we find that for $1 \leq i \leq d$, if i is congruent to 0 modulo the characteristic of $\mathcal{F}$, then $\theta_i = \theta_0$, a contradiction. The result follows.
(iii) Using (88), we find that for any even integer i, $(1 \leq i \leq d)$, if i is congruent to 0 modulo the characteristic of $\mathcal{F}$, then $\theta_i = \theta_0$, a contradiction. The result follows.
(iv) Suppose $d \geq 4$. Applying (89), we find $\theta_0 = \theta_4$, a contradiction.
$\square$

Lemma 9.4 *With reference to Definition 9.1, assume* $\theta_0, \theta_1, \ldots, \theta_d$ *are distinct. Pick any integers* i, j, r, s $(0 \leq i, j, r, s \leq d)$ *and assume* $i + j = r + s$, $r \neq s$. *Then (i)–(v) hold below.*

(i) Suppose $\beta \neq 2$, $\beta \neq -2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} \;=\; \frac{q^i - q^j}{q^r - q^s}, \tag{90}$$

where $q + q^{-1} = \beta$.

(ii) Suppose $\beta = 2$ *and char*$(\mathcal{F}) \neq 2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} \;=\; \frac{i-j}{r-s}. \tag{91}$$

(iii) Suppose $\beta = -2$ *and char*$(\mathcal{F}) \neq 2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} \;=\; \begin{cases} (-1)^{i+r}\frac{i-j}{r-s} & \text{if } i+j \text{ is even,} \\ (-1)^{i+r} & \text{if } i+j \text{ is odd.} \end{cases} \tag{92}$$

(iv) Suppose $\beta = 0$ *and char*$(\mathcal{F}) = 2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} \;=\; \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \tag{93}$$

Proof: To get (i), evaluate the left side in (90) using (86), and simplify the result. The cases (ii)–(iv) are very similar.
□
We finish this section with an observation.

Lemma 9.5 *With the notation and assumptions of Lemma 9.4, the scalar*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s}$$

depends only on i, j, r, s and β, and not on $\theta_0, \theta_1, \ldots, \theta_d$.

Proof: This is immediate from the data in Lemma 9.4.
□

10 A sum

Definition 10.1 *Throughout this section, $\mathcal{F}$ will denote a field, d will denote an integer at least 1, and $\theta_0, \theta_1, \ldots, \theta_d$ will denote a sequence of distinct scalars in $\mathcal{F}$. We let β denote any scalar in $\mathcal{F}$.*

With reference to Definition 10.1, we now consider the sums

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}, \tag{94}$$

where $0 \leq i \leq d+1$. Denoting the sum in (94) by ϑ_i, we remark

$$\vartheta_0 = 0, \qquad \vartheta_1 = 1, \qquad \vartheta_d = 1, \qquad \vartheta_{d+1} = 0. \tag{95}$$

Moreover

$$\vartheta_i = \vartheta_{d-i+1} \qquad (0 \leq i \leq d+1), \tag{96}$$

and

$$\vartheta_{i+1} - \vartheta_i = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d). \tag{97}$$

It turns out the sums (94) play an important role a bit later, so we will examine them carefully. We begin by giving explicit formulae for the sums (94) under the assumption the sequence $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent. To avoid trivialities, we assume $d \geq 3$.

Lemma 10.2 *With reference to Definition 10.1, assume $d \geq 3$, and assume $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent. Then for all integers i $(0 \leq i \leq d+1)$, we have the following.*

(i) Suppose $\beta \neq 2$, $\beta \neq -2$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{(q^i - 1)(q^{d-i+1} - 1)}{(q-1)(q^d - 1)}, \tag{98}$$

where $q + q^{-1} = \beta$.

(ii) Suppose $\beta = 2$ and char($\mathcal{F}$) $\neq 2$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{i(d-i+1)}{d}. \tag{99}$$

(iii) Suppose $\beta = -2$, char($\mathcal{F}$) $\neq 2$, and d odd. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \tag{100}$$

(iv) Suppose $\beta = -2$, char($\mathcal{F}$) $\neq 2$, and d even. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} i/d & \text{if } i \text{ is even} \\ (d-i+1)/d & \text{if } i \text{ is odd.} \end{cases} \tag{101}$$

(v) Suppose $\beta = 0$, char($\mathcal{F}$) $= 2$, and $d = 3$. Then

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \tag{102}$$

Proof: The above sums can be computed directly from Lemma 9.4.
□
We mention some recursions satisfied by the sums (94).

Lemma 10.3 *With reference to Definition 10.1, assume $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent, and put*

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d+1). \tag{103}$$

Then (i),(ii) hold below.

(i) $\vartheta_{i+1} = \vartheta_i \dfrac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + 1 \qquad (1 \leq i \leq d),$

(ii) $\vartheta_i = \vartheta_{i+1} \dfrac{\theta_i - \theta_1}{\theta_{i+1} - \theta_0} + 1 \qquad (0 \leq i \leq d-1).$

Proof: (i) These equations are readily verified case by case, using Lemma 10.2.
(ii) Apply (i) above to the sequence $\theta_d, \theta_{d-1}, \ldots, \theta_0$, and use (96).
□

Lemma 10.4 *With reference to Definition 10.1, assume $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent. Let r denote any integer in the range $1 \leq r \leq d+1$, and suppose we are given scalars $\vartheta_1, \vartheta_2, \ldots, \vartheta_r$ in $\mathcal{F}$ such that*

$$\vartheta_{i+1} = \vartheta_i \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + \vartheta_1 \qquad (1 \leq i \leq r-1). \tag{104}$$

Then

$$\vartheta_i = \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (1 \leq i \leq r). \tag{105}$$

Proof: Define

$$\vartheta'_i = \vartheta_i - \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (106)$$

for $1 \leq i \leq r$, and observe $\vartheta'_1 = 0$. Combining Lemma 10.3(i) and (104), we routinely find

$$\vartheta'_{i+1} = \vartheta'_i \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} \qquad (1 \leq i \leq r-1). \qquad (107)$$

Apparently $\vartheta'_i = 0$ for $1 \leq i \leq r$, and the result follows.
□
We mention an identity that will be useful later.

Lemma 10.5 *With reference to Definition 10.1, assume $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent. Then*

$$\frac{\theta_0 - \theta_1 + \theta_{i-1} - \theta_i}{\theta_0 - \theta_i} \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{\theta_0 + \theta_{i-1} - \theta_{d-i+1} - \theta_d}{\theta_0 - \theta_d}, \qquad (108)$$

for $1 \leq i \leq d$. (Caution: the numerator on the far left in (108) might be zero).

Proof: Add (97) and Lemma 10.3(ii), solve the resulting equation for ϑ_{i+1}, and replace i by $i-1$ in the result.
□
Here is another recursion.

Lemma 10.6 *With reference to Definition 10.1, assume $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent, and put*

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d+1). \qquad (109)$$

Then the sequence $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ is β-recurrent.

Proof: We show

$$\vartheta_{i-2} \;-\; (\beta+1)\vartheta_{i-1} \;+\; (\beta+1)\vartheta_i \;-\; \vartheta_{i+1} \qquad (110)$$

is zero for $2 \leq i \leq d$. First observe by (84) that

$$\theta_{j-1} - \beta\theta_j + \theta_{j+1} \;=\; \theta_{d-j-1} - \beta\theta_{d-j} + \theta_{d-j+1} \qquad (1 \leq j \leq d-1). \qquad (111)$$

Eliminating ϑ_{i-2}, ϑ_{i-1}, ϑ_i, ϑ_{i+1} in (110) using (109), then cancelling terms where possible, and then simplifying the result using (111), we get zero.
□
For completness sake, we include a lemma concerning the converse to Lemma 10.6. We do not use the result, so we will not dwell on the proof.

Lemma 10.7 *With reference to Definition 10.1, assume $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent. Let $\vartheta_0, \vartheta_1, \ldots,$ ϑ_{d+1} denote a β-recurrent sequence of scalars taken from $\mathcal{F}$, such that $\vartheta_0 = 0$, $\vartheta_{d+1} = 0$, and $\vartheta_1 = \vartheta_d$. Then*

$$\vartheta_i = \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d+1).$$

Proof: Routine calculation using Lemma 9.2, Lemma 9.3, and Lemma 10.2.
□

11 Some equations involving the split canonical form

In this section, we return to the situation of Definition 4.1, and determine when the products $E_d A^* E_i$ vanish for $0 \le i \le d-2$. We begin with a definition.

Definition 11.1 *With reference to Definition 4.1, we define*

$$\vartheta_i = \varphi_i - (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \le i \le d), \tag{112}$$

and $\vartheta_0 = 0$, $\vartheta_{d+1} = 0$. We observe ϑ_1 equals the scalar ϕ_1 from Definition 7.1.

Our goal in this section is to prove the following theorem.

Theorem 11.2 *With reference to Definition 4.1, assume $d \ge 2$. Then the following are equivalent.*

(i) $E_d A^* E_i = 0 \qquad (0 \le i \le d-2).$

(ii) $\vartheta_{i+1} = \vartheta_i \dfrac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + \vartheta_1 \qquad (1 \le i \le d-1),$

where $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ are from (112).

To prove the above theorem, it is advantageous to consider the linear combination

$$\sum_{i=0}^{d-2} E_d A^* E_i (\theta_i - \theta_{d-1}).$$

Lemma 11.3 *With reference to Definition 4.1, assume $d \ge 2$. Then*

$$\sum_{i=0}^{d-2} E_d A^* E_i (\theta_i - \theta_{d-1}) = E_d (A^* - a^*_d I)(A - \theta_{d-1} I), \tag{113}$$

where the scalar a^*_d is from Definition 2.5.

Proof: Since E_d is a rank one idempotent, we find $E_d A^* E_d$ is a scalar multiple of E_d. Taking the trace, we find

$$E_d A^* E_d = a^*_d E_d. \tag{114}$$

We may now argue

$$\begin{aligned}
&E_d (A^* - a^*_d I)(A - \theta_{d-1} I) \\
&\quad = E_d (A^* - a^*_d I)(A - \theta_{d-1} I) \sum_{i=0}^{d} E_i && \text{(by (4))} \\
&\quad = \sum_{i=0}^{d} E_d (A^* - a^*_d I) E_i (\theta_i - \theta_{d-1}) && \text{(by (2))} \\
&\quad = \sum_{i=0}^{d-2} E_d (A^* - a^*_d I) E_i (\theta_i - \theta_{d-1}) && \text{(by (114))} \\
&\quad = \sum_{i=0}^{d-2} E_d A^* E_i (\theta_i - \theta_{d-1}) && \text{(by (3))},
\end{aligned}$$

as desired.
□

Corollary 11.4 *With reference to Definition 4.1, assume $d \geq 2$, and let the scalar a_d^* be as in Definition 2.5. Then the following are equivalent.*

(i) $E_d A^* E_i = 0 \qquad (0 \leq i \leq d-2)$.

(ii) $E_d(A^* - a_d^* I)(A - \theta_{d-1} I) = 0$.

Proof: $(i) \to (ii)$ Immediate from Lemma 11.3.
$(ii) \to (i)$ Multiply both sides of (113) on the right by each $E_0, E_1, \ldots, E_{d-2}$, and simplify using (3).
□

Lemma 11.5 *With reference to Definition 4.1, assume $d \geq 2$, and consider the matrix*

$$E_d(A^* - a_d^* I)(A - \theta_{d-1} I), \tag{115}$$

where a_d^* is from Definition 2.5. For the matrix (115), all entries in rows $0, 1, \ldots, d-1$ are zero. The entries in the d^{th} row of (115) are as follows. For $0 \leq i \leq d$, the di^{th} entry of (115) is $\tau_i(\theta_d)\tau_d(\theta_d)^{-1}$ times

$$\vartheta_{i+1} \;-\; \vartheta_i \, \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} \;-\; \vartheta_d, \tag{116}$$

where the ϑ_j are from Definition 11.1, and where θ_{-1} is an indeterminant.

Proof: To obtain our first assertion, observe by (53) that in the matrix E_d, all entries in rows $0, 1, \ldots, d-1$ are zero. E_d is the factor on the left in (115). By matrix multiplication we see that in (115), all entries in rows $0, 1, \ldots, d-1$ are zero. We now consider the d^{th} row of (115). To compute it, we recall the d^{th} row of E_d. By (53), we find the entries

$$\begin{aligned} (E_d)_{di} &= \frac{\tau_i(\theta_d)}{\tau_d(\theta_d)} \\ &= \frac{1}{(\theta_d - \theta_i)(\theta_d - \theta_{i+1}) \cdots (\theta_d - \theta_{d-1})} \end{aligned}$$

for $0 \leq i \leq d$. Multiplying out (115) using this, we routinely find its di^{th} entry is $\tau_i(\theta_d)\tau_d(\theta_d)^{-1}$ times

$$(\theta_i^* - a_d^*)(\theta_i - \theta_{d-1}) \;+\; (\theta_{i+1}^* - a_d^*)(\theta_d - \theta_i) \;+\; \varphi_{i+1} \;-\; \varphi_i \, \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d}$$

for $0 \leq i \leq d$. Eliminating a_d^* in the above line using (68), and eliminating φ_i, φ_{i+1}, φ_d in the result using (112), we obtain (116).
□

Proof of Theorem 11.2: $(i) \to (ii)$ Apparently Corollary 11.4(i) holds, so Corollary 11.4(ii) holds, and the matrix (115) is zero. Applying Lemma 11.5, we find the expressions (116) are zero for $0 \leq i \leq d$. Setting $i = 0$, $\vartheta_0 = 0$ in (116), we find $\vartheta_1 = \vartheta_d$. Using this to eliminate ϑ_d in (116), we obtain the equations in Theorem 11.2(ii).
$(ii) \to (i)$ Setting $i = d-1$ in the given equations, we find $\vartheta_d = \vartheta_1$. Using this to eliminate ϑ_1 on the right in the remaining given equations, we find the expressions (116) are zero for $1 \leq i \leq d-1$. We routinely find the expression (116) is zero for $i = 0$ and $i = d$, so (116) is zero for $0 \leq i \leq d$. Applying Lemma 11.5, we find the matrix (115) is zero. Applying Corollary 11.4, we find $E_d A^* E_i$ vanishes for $0 \leq i \leq d-2$.
□

12 Two polynomial equations for A and A^*

In this section, we show the elements A and A^* in a Leonard pair satisfy two cubic polynomial equations. We begin with a comment on the situation of Definition 2.1.

Lemma 12.1 *With reference to Definition 2.1, suppose*

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j| > 1, \qquad\qquad (0 \le i, j \le d), \tag{117}$$

and let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. Then

$$\operatorname{Span}\{XA^*Y - YA^*X \mid X, Y \in \mathcal{D}\} = \{XA^* - A^*X \mid X \in \mathcal{D}\}.$$

Proof: For notational convenience set $E_{-1} = 0$, $E_{d+1} = 0$. We claim that for $0 \le i \le d$,

$$E_i A^* E_{i+1} - E_{i+1} A^* E_i = L_i A^* - A^* L_i, \tag{118}$$

where $L_i := E_0 + E_1 + \cdots + E_i$. To see (118), observe by (4) and (117) that for $0 \le j \le d$, both

$$E_j A^* = E_j A^* E_{j-1} + E_j A^* E_j + E_j A^* E_{j+1}, \tag{119}$$

$$A^* E_j = E_{j-1} A^* E_j + E_j A^* E_j + E_{j+1} A^* E_j. \tag{120}$$

Summing (119) over $j = 0, 1, \ldots, i$, summing (120) over $j = 0, 1, \ldots, i$, and taking the difference between the two sums, we obtain (118). Observe $\mathcal{D}$ is spanned by both $E_0, E_1, \ldots, E_d$ and $L_0, L_1, \ldots, L_d$, so

$$\begin{aligned}
&\operatorname{Span}\{XA^*Y - YA^*X \mid X, Y \in \mathcal{D}\} \\
&\qquad = \operatorname{Span}\{E_i A^* E_j - E_j A^* E_i \mid 0 \le i, j \le d\} \\
&\qquad = \operatorname{Span}\{E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \le i \le d\} \\
&\qquad = \operatorname{Span}\{L_i A^* - A^* L_i \mid 0 \le i \le d\} \\
&\qquad = \{XA^* - A^*X \mid X \in \mathcal{D}\},
\end{aligned}$$

and we are done.
$\square$
We now assume the situation of Definition 4.1, and consider the implications of Lemma 12.1.

Lemma 12.2 *With reference to Definition 4.1, assume*

$$E_i A^* E_j = 0 \quad \text{if} \quad i - j > 1, \qquad\qquad (0 \le i, j \le d). \tag{121}$$

Then there exists scalars β, γ, ϱ in $\mathcal{F}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \tag{122}$$

where $[r, s]$ means $rs - sr$.

Proof: First assume $d \ge 3$. Combining (121) and Lemma 4.9(i), we obtain (117); applying Lemma 12.1, we find there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_d$ in $\mathcal{F}$ such that

$$A^2A^*A - AA^*A^2 = \sum_{i=1}^{d} \alpha_i (A^i A^* - A^* A^i). \tag{123}$$

We show $\alpha_i = 0$ for $4 \le i \le d$. Suppose not, and set

$$t := \max\{i \mid 4 \le i \le d, \ \ \alpha_i \ne 0\}.$$

Computing the $t0$ entry of each term in (123), we readily find

$$0 = \alpha_t(\theta^*_0 - \theta^*_t),$$

an impossibility. We now have $\alpha_i = 0$ for $4 \le i \le d$, so (123) becomes

$$\begin{aligned} A^2A^*A - AA^*A^2 = \alpha_1(AA^* - A^*A) \ + \ \alpha_2(A^2A^* - A^*A^2) \\ + \alpha_3(A^3A^* - A^*A^3). \end{aligned} \tag{124}$$

We show $\alpha_3 \ne 0$. Suppose $\alpha_3 = 0$. Computing the 30 entry in (124), we readily find $\theta^*_1 - \theta^*_2 = 0$, an impossibility. We have now shown $\alpha_3 \ne 0$. Set

$$\begin{aligned} C := \alpha_1 A^* \ + \ \alpha_2(AA^* + A^*A) \ + \ \alpha_3(A^2A^* + A^*A^2) \\ + (\alpha_3 - 1)AA^*A. \end{aligned} \tag{125}$$

Observe $AC - CA$ equals

$$\begin{aligned} \alpha_1(AA^* - A^*A) \ + \ \alpha_2(A^2A^* - A^*A^2) \ + \ \alpha_3(A^3A^* - A^*A^3) \\ + AA^*A^2 - A^2A^*A, \end{aligned}$$

and this equals 0 in view of (124). Hence A and C commute. Dividing C by α_3 and using (125), we find A commutes with

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*,$$

where

$$\beta := \alpha_3^{-1} - 1, \qquad \gamma := -\alpha_2\alpha_3^{-1}, \qquad \varrho := -\alpha_1\alpha_3^{-1}.$$

We now have (122) for the case $d \ge 3$. For $d \le 2$, we adjust our argument a bit. Let α_3 denote any nonzero scalar in $\mathcal{F}$. By our initial comments, and since A^3 is a linear combination of I, A, A^2, we find there exists scalars α_1, α_2 in $\mathcal{F}$ such that (124) holds. Proceeding as before, we obtain (122). $\square$

Concerning the converse to Lemma 12.2, we have the following.

Lemma 12.3 *With reference to Definition 4.1, suppose there exists scalars β, γ, ϱ in $\mathcal{F}$ such that*

$$0 \quad = \quad [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*]. \tag{126}$$

Then

$$E_iA^*E_j \quad = \quad 0 \quad \text{if} \quad 1 < i - j < d, \qquad\qquad (0 \le i, j \le d). \tag{127}$$

Proof: We define a two variable polynomial $p \in \mathcal{F}[\lambda, \mu]$ by

$$p(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.$$

We claim

$$0 = E_iA^*E_jp(\theta_i, \theta_j) \quad \text{if} \quad i \ne j, \qquad\qquad (0 \le i, j \le d). \tag{128}$$

To prove this, set

$$C \quad = \quad A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*,$$

so that $AC = CA$. For $0 \leq i, j \leq d$,

$$\begin{aligned} 0 &= E_i(AC - CA)E_j \\ &= (\theta_i - \theta_j)E_iCE_j, \end{aligned}$$

so if $i \neq j$,

$$\begin{aligned} 0 &= E_iCE_j \\ &= E_iA^*E_j(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \varrho) \\ &= E_iA^*E_jp(\theta_i, \theta_j), \end{aligned}$$

and we have (128). We next claim

$$p(\theta_i, \theta_j) \neq 0 \quad \text{if} \quad 1 < i - j < d, \qquad\qquad (0 \leq i, j \leq d). \tag{129}$$

Recall $E_{i-1}A^*E_i \neq 0$ for $1 \leq i \leq d$ by Lemma 4.9(i). By this and (128), we find $p(\theta_{i-1}, \theta_i) = 0$ for $1 \leq i \leq d$. Since p is symmetric in its arguments, we find θ_{i-1} and θ_{i+1} are the roots of $p(\lambda, \theta_i)$ for $1 \leq i \leq d-1$. By this and since $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, we obtain (129). Combining (128) and (129) we obtain (127).
$\square$

Lemma 12.4 *With reference to Definition 4.1, let β, γ, ϱ denote any scalars in $\mathcal{F}$, and consider the commutator*

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*]. \tag{130}$$

Then the entries of (130) are as follows.

(i) The $i+1, i-2$ entry is

$$\theta^*_{i-2} \;-\; (\beta+1)\theta^*_{i-1} \;+\; (\beta+1)\theta^*_i \;-\; \theta^*_{i+1},$$

for $2 \leq i \leq d-1$.

(ii) The $i, i-2$ entry is

$$\begin{aligned} &\vartheta_{i-2} \;-\; (\beta+1)\vartheta_{i-1} \;+\; (\beta+1)\vartheta_i \;-\; \vartheta_{i+1} \\ &+ (\theta^*_{i-2} - \theta^*_0)(\theta_{i-3} \;-\; (\beta+1)\theta_{i-2} \;+\; (\beta+1)\theta_{i-1} \;-\; \theta_i) \\ &+ (\theta_i - \theta_d)(\theta^*_{i-2} \;-\; (\beta+1)\theta^*_{i-1} \;+\; (\beta+1)\theta^*_i \;-\; \theta^*_{i+1}) \\ &+ (\theta^*_{i-2} - \theta^*_i)(\theta_{i-2} \;-\; \beta\theta_{i-1} \;+\; \theta_i \;-\; \gamma) \end{aligned}$$

for $2 \leq i \leq d$, where $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ are from Definition 11.1.

(iii) The $i, i-1$ entry is

$$\begin{aligned} &\varphi_{i-1}(\theta_{i-2} - \beta\theta_{i-1} + \theta_i - \gamma) \;-\; \varphi_{i+1}(\theta_{i-1} - \beta\theta_i + \theta_{i+1} - \gamma) \\ &+ (\theta^*_{i-1} - \theta^*_i)(\theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i) - \varrho), \end{aligned}$$

for $1 \leq i \leq d$.

(iv) The ii entry is

$$\varphi_i(\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) - \varrho)$$
$$- \varphi_{i+1}(\theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \varrho),$$

for $0 \leq i \leq d$.

(v) The $i-1, i$ *entry is*

$$\varphi_i(\theta_{i-1} - \theta_i)(\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) - \varrho),$$

for $1 \leq i \leq d$.

All remaining entries in (130) are zero. In the above formulae, we assume $\varphi_0 = 0$, $\varphi_{d+1} = 0$, *and that* θ_{-1}, θ_{d+1}, θ^*_{d+1} *are indeterminants.*

Proof: Routine matrix multiplication.
□

Theorem 12.5 *With reference to Definition 4.1, let* β, γ, ϱ *denote any scalars in* $\mathcal{F}$. *Then*

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] \tag{131}$$

if and only if (i)–(iii) hold below.

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ, ϱ)*-recurrent.*

(ii) The sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ *is* β*-recurrent.*

(iii) The sequence $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ *from Definition 11.1 is* β*-recurrent.*

Proof: First assume (131). Then (130) is zero, so all its entries given in Lemma 12.4 are zero. In particular, for $1 \leq i \leq d$, the expression in Lemma 12.4(v) is zero. In that expression the two factors on the left are nonzero, so the remaining factor

$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) - \varrho$$

is zero. Now $\theta_0, \theta_1, \ldots, \theta_d$ is (β, γ, ϱ)-recurrent by Definition 8.2(iv). For $2 \leq i \leq d-1$, the expresssion in Lemma 12.4(i) is zero, so $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ is β-recurrent by Definition 8.2(ii). For $2 \leq i \leq d$, the expression in Lemma 12.4(ii) is zero. Consider the four lines in that expression. The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is (β, γ)-recurrent by Lemma 8.5, so line 4 is zero. The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent by Lemma 8.4, so line 2 is zero. We mentioned $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ is β-recurrent, so line 3 is zero. Apparently line 1 is zero, so $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ is β-recurrent in view of Definition 8.2(ii). We are now done in one direction. To get the converse, suppose (i)–(iii) hold. Applying Lemma 8.4 and Lemma 8.5, we find $\theta_0, \theta_1, \ldots, \theta_d$ is both β-recurrent and (β, γ)-recurrent. From these facts and the data in Lemma 12.4, we find all the entries of (130) are zero, so (130) is zero. We now have (131).
□
Modifying our point of view in Theorem 12.5, we get the following result.

Corollary 12.6 *With reference to Definition 4.1, let β denote any scalar in $\mathcal{F}$. Then there exists scalars γ, ϱ in $\mathcal{F}$ such that*

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] \tag{132}$$

if and only if (i)–(iii) hold below.

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent.

(ii) The sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ is β-recurrent.

(iii) The sequence $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ from Definition 11.1 is β-recurrent.

Proof: Recall $\theta_0, \theta_1, \ldots, \theta_d$ are distinct by Definition 4.1. Applying Lemma 8.4 and Lemma 8.5, we find $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent if and only if there exists $\gamma, \varrho \in \mathcal{F}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is (β, γ, ϱ)-recurrent. The result now follows in view of Theorem 12.5.
□
We now have enough information to obtain our classification theorem in one direction.

Lemma 12.7 *Let Φ denote a Leonard system with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$. Then (i)–(iii) hold below.*

(i) $$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \; + \; (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d),$$

(ii) $$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \; + \; (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d),$$

(iii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{133}$$

are equal and independent of i, for $2 \leq i \leq d-1$.

Proof: We write $\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$, and begin by proving (iii).
(iii). Applying Lemma 12.2 to the split canonical form of Φ, we find there exists scalars β, γ, ϱ such that (122) holds. Applying Corollary 12.6, we find both the eigenvalue sequence and the dual eigenvalue sequence of Φ are β-recurrent. Using this, we find the expressions (133) equal $\beta + 1$ for $2 \leq i \leq d-1$. In particular, these expressions are equal and independent of i.
(i). We first claim

$$\vartheta_i = \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (1 \leq i \leq d), \tag{134}$$

where $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ are from Definition 11.1. To see (134), we combine Lemma 10.4 and Theorem 11.2. Assume $d \geq 2$; otherwise (134) is trivial. Observe $E_dA^*E_i = 0$ for $0 \leq i \leq d-2$; applying Theorem 11.2 to the split canonical form of Φ, we find

$$\vartheta_{i+1} = \vartheta_i \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + \vartheta_1 \qquad (1 \leq i \leq d-1). \tag{135}$$

By (135), and since $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent, we obtain the assumptions of Lemma 10.4 (with $r = d$). Applying that lemma, we obtain (134). In (134), we eliminate ϑ_i on the left using Definition 11.1, and set $\vartheta_1 = \phi_1$ on the right, to get the result.
(ii). Apply (i) above to $\Phi^{\Downarrow}$, and use Theorem 1.11.
□
We finish this section by proving Theorem 1.12 from the Introduction.

Proof of Theorem 1.12: Let Φ denote a Leonard system associated with (A, A^*), and abbreviate $\mathcal{A} = \mathcal{A}(\Phi)$. If the diameter $d \leq 2$, then any pair of multiplicity-free elements A, A^* from $\mathcal{A}$ satisfy (16), (17), if we choose $\beta = -1$ and appropriate γ, γ^*, ϱ, ϱ^*. For the rest of the proof, assume $d \geq 3$. Applying Lemma 12.2 to the split canonical form of Φ, we find there exist scalars β, γ, ϱ in $\mathcal{F}$ such that (16) holds. Applying the above argument to Φ^*, we find there exist scalars $\beta^*, \gamma^*, \varrho^*$ in $\mathcal{F}$ such that

$$0 = [A^*, A^{*2}A - \beta^* A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A].$$

We show $\beta^* = \beta$. Applying Corollary 12.6 to the split canonical form of Φ, we find both the eigenvalue sequence and dual eigenvalue sequence of Φ are β-recurrent. From this we find $\beta + 1$ equals the common value of (133). Applying this argument to Φ^*, we find $\beta^* + 1$ also equals the common value of (133), so $\beta = \beta^*$. We now have (17). Concerning uniqueness, we showed $\beta + 1$ equals the common value of (133), so β is uniquely determined by (A, A^*). Applying Theorem 12.5 to the split canonical form of Φ, we find the eigenvalue sequence is (β, γ, ϱ)-recurrent, so γ, ϱ are determined by this sequence. Applying this argument to Φ^*, we find the dual eigenvalue sequence is $(\beta, \gamma^*, \varrho^*)$-recurrent, so γ^*, ϱ^* are determined by this sequence.
□

13 Some vanishing products

In this section, we establish a few facts that we need to complete the proof of the classification theorem.

Definition 13.1 *In this section, we assume we are in the situation of Definition 4.1, and we further assume (i), (ii) below.*

(i) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d).$

(ii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{136}$$

are equal and independent of i, *for* $2 \leq i \leq d - 1$.

(the scalar ϕ_1 *is from (79)).*

Lemma 13.2 *With reference to Definition 13.1, lines (i), (ii) hold below.*

(i) $E_i A^* E_j = 0 \quad \text{if} \quad i - j > 1, \qquad (0 \leq i, j \leq d).$

(ii) $E^*_i A E^*_j = 0 \quad \text{if} \quad j - i > 1, \qquad (0 \leq i, j \leq d).$

Proof: (i) We assume $d \geq 2$; otherwise there is nothing to prove. We first claim

$$\vartheta_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \tag{137}$$

for $0 \leq i \leq d+1$, where $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ are from Definition 11.1. To obtain (137) for $1 \leq i \leq d$, eliminate φ_i in Definition 13.1(i) using (112). Line (137) holds for $i = 0$ and $i = d+1$, since in these cases both sides of (137) are zero. We proceed in two steps. We first show

$$E_i A^* E_j \quad = \quad 0 \quad \text{if} \quad 1 < i - j < d, \qquad\qquad (0 \leq i, j \leq d). \tag{138}$$

To do this, we apply Lemma 12.3. By Definition 13.1(ii), there exists $\beta \in \mathcal{F}$ such that both $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are β-recurrent. Now by (137) and Lemma 10.6, we find $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ is β-recurrent. Now conditions (i)–(iii) hold in Corollary 12.6. Applying that corollary, we find there exists scalars γ, ϱ in $\mathcal{F}$ such that (132) holds. Applying Lemma 12.3, we obtain (138). To remove the restriction $i - j < d$ in (138), we show $E_d A^* E_0 = 0$. By (137), Lemma 10.3(i), and since $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent, we find

$$\vartheta_{i+1} = \vartheta_i \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + \vartheta_1 \qquad\qquad (1 \leq i \leq d). \tag{139}$$

In particular Theorem 11.2(ii) holds. Applying that theorem, we find $E_d A^* E_i = 0$ for $0 \leq i \leq d-2$, and in particular $E_d A^* E_0 = 0$.

(ii) Consider the matrices $A' := ZGA^*G^{-1}Z$ and $A^{*\prime} := ZGAG^{-1}Z$ from Lemma 4.2(iii). From that lemma, we observe $A', A^{*\prime}$ satisfy the conditions of Definition 4.1. We show $A', A^{*\prime}$ satisfy the conditions (i), (ii) of Definition 13.1. To this end, define

$$\theta'_i := \theta^*_{d-i}, \qquad\qquad \theta^{*\prime}_i := \theta_{d-i} \qquad\qquad (0 \leq i \leq d), \tag{140}$$

$$\varphi'_i := \varphi_{d-i+1} \qquad\qquad (1 \leq i \leq d), \tag{141}$$

and put

$$\phi'_1 := \varphi'_1 - (\theta^{*\prime}_1 - \theta^{*\prime}_0)(\theta'_0 - \theta'_d) \tag{142}$$

in view of (79). We show

$$\varphi'_i = \phi'_1 \sum_{h=0}^{i-1} \frac{\theta'_h - \theta'_{d-h}}{\theta'_0 - \theta'_d} \;+\; (\theta^{*\prime}_i - \theta^{*\prime}_0)(\theta'_{i-1} - \theta'_d) \tag{143}$$

for $1 \leq i \leq d$. Assume $d \geq 1$, and let i be given. By Definition 13.1(i),

$$\varphi_d = \phi_1 + (\theta^*_d - \theta^*_0)(\theta_{d-1} - \theta_d). \tag{144}$$

Evaluating the right side of (142) using (140), (141), (144), we obtain

$$\phi'_1 = \phi_1. \tag{145}$$

By Lemma 9.5 and Definition 13.1(ii),

$$\frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{\theta^*_h - \theta^*_{d-h}}{\theta^*_0 - \theta^*_d} \qquad\qquad (0 \leq h \leq d). \tag{146}$$

By (96), (140), and (146),

$$\sum_{h=0}^{i-1} \frac{\theta'_h - \theta'_{d-h}}{\theta'_0 - \theta'_d} = \sum_{h=0}^{d-i} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}. \tag{147}$$

The right side of (143), upon simplification using (140), (145), and (147), becomes

$$\phi_1 \sum_{h=0}^{d-i} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_{d-i} - \theta_d)(\theta^*_{d-i+1} - \theta^*_0). \tag{148}$$

Replacing i by $d-i+1$ in Definition 13.1(i), we find (148) equals φ_{d-i+1}. Recall $\varphi_{d-i+1} = \varphi'_i$ by (141), so (143) holds. It is clear Definition 13.1(ii) holds after we replace θ_j, θ^*_j by $\theta'_j, \theta^{*\prime}_j$ for $0 \leq j \leq d$. We have now shown $A', A^{*\prime}$ satisfy the conditions (i), (ii) of Definition 13.1, so we can apply part (i) of the present lemma to that pair. For $0 \leq i \leq d$, let E'_i denote the primitive idempotent of A' associated with θ'_i, and observe

$$E'_i = ZGE^*_{d-i}G^{-1}Z \qquad (0 \leq i \leq d). \tag{149}$$

By part (i) of the present lemma,

$$E'_i A^{*\prime} E'_j = 0 \quad \text{if} \quad i - j > 1, \qquad (0 \leq i, j \leq d). \tag{150}$$

Evaluating (150) using (149) and the definition of $A^{*\prime}$, we obtain

$$E^*_{d-i} A E^*_{d-j} = 0 \quad \text{if} \quad i - j > 1, \qquad (0 \leq i, j \leq d).$$

Replacing i and j in the above line by $d-i$ and $d-j$, respectively, we obtain

$$E^*_i A E^*_j = 0 \quad \text{if} \quad j - i > 1, \qquad (0 \leq i, j \leq d).$$

□

Lemma 13.3 *With reference to Definition 13.1, the scalars $\phi_1, \phi_2, \ldots, \phi_d$ from (79) are given by*

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d). \tag{151}$$

Proof: Assume $d \geq 1$; otherwise there is nothing to prove. By Lemma 9.5, Lemma 10.5, and Definition 13.1(ii),

$$\frac{\theta^*_0 - \theta^*_1 + \theta^*_{i-1} - \theta^*_i}{\theta^*_0 - \theta^*_i} \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{\theta_0 + \theta_{i-1} - \theta_{d-i+1} - \theta_d}{\theta_0 - \theta_d}. \tag{152}$$

By (79),

$$\phi_1 \quad = \quad \varphi_1 - (\theta^*_1 - \theta^*_0)(\theta_0 - \theta_d). \tag{153}$$

Adding (79) to the equation in Definition 13.1(i), and simplifying the result using (152), (153), we routinely obtain (151).
□

Lemma 13.4 *With reference to Definition 13.1, suppose the scalars $\phi_1, \phi_2, \ldots, \phi_d$ from (79) are all nonzero. Then the products $E_i A^* E_{i-1}$ and $E^*_{i-1} A E^*_i$ are nonzero for $1 \le i \le d$.*

Proof: Let $V = \mathcal{F}^{d+1}$ denote the irreducible left module for $\mathrm{Mat}_{d+1}(\mathcal{F})$. By Theorem 7.3, and since each of $\phi_1, \phi_2, \ldots, \phi_d$ is nonzero, we find V is irreducible as an (A, A^*)-module. Suppose there exists an integer i $(1 \le i \le d)$ such that $E_i A^* E_{i-1} = 0$, and consider the sum

$$E_0 V + E_1 V + \cdots + E_{i-1} V. \tag{154}$$

Applying Lemma 2.3 to the set $S = \{0, 1, \ldots, i-1\}$, and using Lemma 13.2(i), we find (154) is an (A, A^*)-module. The module (154) is not 0 or V by (5), and since $1 \le i \le d$. This contradicts our above comment that V is irreducible as an (A, A^*)-module, so we conclude $E_i A^* E_{i-1} \neq 0$ for $1 \le i \le d$. Next suppose there exists an integer i $(1 \le i \le d)$ such that $E^*_{i-1} A E^*_i = 0$, and consider the sum

$$E^*_i V + E^*_{i+1} V + \cdots + E^*_d V. \tag{155}$$

Applying Lemma 2.4 to the set $S^* = \{i, i+1, \ldots, d\}$, and using Lemma 13.2(ii), we find (155) is an (A, A^*)-module. We observe (155) is not 0 or V, contradicting the fact that V is irreducible as an (A, A^*)-module. We conclude $E^*_{i-1} A E^*_i \neq 0$ for $1 \le i \le d$.
□

14 A classification of Leonard systems

We are now ready to prove our classification theorem for Leonard systems, which is Theorem 1.9 from the Introduction.

Proof of Theorem 1.9: To prove the theorem in one direction, let

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$$

denote a Leonard system over $\mathcal{F}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$. We verify conditions (i)–(v) in the statement of the theorem. Condition (i) holds by Definition 3.10, Definition 3.12, and the last assertion of Theorem 3.2. Condition (ii) holds by Definition 1.8, and since A and A^* are multiplicity-free. Conditions (iii)–(v) are immediate from Lemma 12.7, and we are done in one direction.

To obtain the converse, suppose (i)–(v) hold in the present theorem, and put

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

We observe A (resp. A^*) is multiplicity-free, with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$, (resp. $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$). For $0 \le i \le d$, let E_i (resp. E^*_i) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ^*_i). We show

$$\Phi := (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d) \tag{156}$$

is a Leonard system in $\mathrm{Mat}_{d+1}(\mathcal{F})$. To do this, we show Φ satisfies the conditions (i)–(v) of Definition 1.4. Conditions (i)–(iii) are clearly satisfied, so consider conditions (iv), (v). By Lemma 4.9,

$$E_i A^* E_j = \begin{cases} 0 & \text{if } \; j-i>1; \\ \neq 0 & \text{if } \; j-i=1 \end{cases} \qquad (0 \leq i,j \leq d) \tag{157}$$

and

$$E^*_i A E^*_j = \begin{cases} 0 & \text{if } \; i-j>1; \\ \neq 0 & \text{if } \; i-j=1 \end{cases} \qquad (0 \leq i,j \leq d). \tag{158}$$

By assumption (iv), the scalar ϕ_1 in the present theorem equals

$$\varphi_1 - (\theta^*_1 - \theta^*_0)(\theta_0 - \theta_d),$$

and therefore equals the scalar denoted ϕ_1 in Definition 7.1. Combining this with assumptions (iii), (v) in the present theorem, we find A and A^* satisfy conditions (i), (ii) of Definition 13.1. Applying Lemma 13.2, we find

$$E_i A^* E_j = 0 \quad \text{if} \quad i-j>1, \qquad (0 \leq i,j \leq d), \tag{159}$$

$$E^*_i A E^*_j = 0 \quad \text{if} \quad j-i>1, \qquad (0 \leq i,j \leq d). \tag{160}$$

By assumption (iv) and Lemma 13.3, the sequence $\phi_1, \phi_2, \ldots, \phi_d$ in the present theorem equals the corresponding sequence from Definition 7.1. The elements of this sequence are nonzero by assumption (i), so by Lemma 13.4,

$$E_i A^* E_j \neq 0 \quad \text{if} \quad i-j=1 \qquad (0 \leq i,j \leq d), \tag{161}$$

$$E^*_i A E^*_j \neq 0 \quad \text{if} \quad j-i=1 \qquad (0 \leq i,j \leq d). \tag{162}$$

Combining (157)–(162), we obtain conditions (iv), (v) of Definition 1.4, so Φ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathcal{F})$. By Lemma 4.10, we find Φ has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, and φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$. We mentioned the sequence $\phi_1, \phi_2, \ldots, \phi_d$ from the present theorem is the same as the corresponding sequence from Definition 7.1, so this is the ϕ-sequence of Φ in view of Lemma 6.5(i). The Leonard system Φ is unique up to isomorphism by Lemma 3.11.
□

Corollary 14.1 *Let Φ denote a Leonard system over $\mathcal{F}$ with diameter $d \geq 3$, eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$. Consider a sequence $\mathcal{S}$ of 9 parameters consisting of the sequence in (i) below, followed by either parameter in (ii) below, followed by any one of the parameters in (iii) below:*

(i) $d, \theta_0, \theta_1, \theta_2, \theta^*_0, \theta^*_1, \theta^*_2$,

(ii) θ_3, θ^*_3,

(iii) $\varphi_1, \phi_1, \varphi_d, \phi_d$.

Then the isomorphism class of Φ as a Leonard system over $\mathcal{F}$ is determined by $\mathcal{S}$.

Proof: From Theorem 1.9(v), we recursively obtain θ_i, θ^*_i for $0 \leq i \leq d$. Using Theorem 1.9(iii),(iv) (with $i = 1$, $i = d$), we obtain ϕ_1. Using Theorem 1.9(iii), we obtain φ_i for $1 \leq i \leq d$. Applying Lemma 3.11, we find the isomorphism class of Φ is determined by $\mathcal{S}$.
□

Corollary 14.2 *Let d denote a nonnegative integer, let $\mathcal{F}$ denote a field, and let A and A^* denote matrices in $Mat_{d+1}(\mathcal{F})$ of the form*

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

Then the following are equivalent.

(i) (A, A^*) *is a Leonard pair in $Mat_{d+1}(\mathcal{F})$.*

(ii) There exist a sequence of scalars $\phi_1, \phi_2, \ldots, \phi_d$ taken from $\mathcal{F}$ such that (i)–(v) hold in Theorem 1.9.

Suppose (i),(ii) hold above. Then

$$(A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d) \tag{163}$$

is a Leonard system in $Mat_{d+1}(\mathcal{F})$, where E_i (resp. E^*_i) denotes the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ^*_i), for $0 \leq i \leq d$. The Leonard system (163) has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$.

Proof: *(i)* $\rightarrow$ *(ii)*. We first show (163) is a Leonard system. To do this, we apply Lemma 4.10. In order to do that, we verify A and A^* satisfy the conditions of Definition 4.1. Certainly $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since A is multiplicity-free. Similarly $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are distinct. Observe $\varphi_1, \varphi_2, \ldots, \varphi_d$ are nonzero; otherwise the left module $V = \mathcal{F}^{d+1}$ of $\text{Mat}_{d+1}(\mathcal{F})$ is reducible as an (A, A^*)-module, contradicting Lemma 3.3. We have now shown A and A^* satisfy the conditions of Definition 4.1, so we can apply Lemma 4.10. By that lemma, we find (163) is a Leonard system, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, and φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$. Let $\phi_1, \phi_2, \ldots \phi_d$ denote the ϕ-sequence of the Leonard system (163). Applying Theorem 1.9 to this system, we find (i)–(v) hold in that theorem.
(ii) $\rightarrow$ *(i)*. By Theorem 1.9, there exists a Leonard system Φ over $\mathcal{F}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, φ-sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and ϕ-sequence $\phi_1, \phi_2, \ldots, \phi_d$. The Leonard system Φ has split canonical form (163) by Definition 3.10, so (163) is a Leonard system in $\text{Mat}_{d+1}(\mathcal{F})$. In particular (A, A^*) is a Leonard pair in $\text{Mat}_{d+1}(\mathcal{F})$, as desired.
Suppose (i),(ii). From the proof of *(ii)* $\rightarrow$ *(i)* we find (163) is a Leonard system in $\text{Mat}_{d+1}(\mathcal{F})$, with the required eigenvalue, dual eigenvalue, φ- and ϕ-sequences.
□

15 Appendix: Leonard systems and polynomials

There is a theorem due to Doug Leonard [27], [5, p260] that gives a characterization of the q-Racah polynomials and some related polynomials in the Askey scheme [2], [3], [4], [25], [26]. The situation considered in that theorem is closely connected to the subject of the present paper, and it is this connection that motivates our terminology. We sketch the connection here without proof; details will be provided in a future paper.

Let $\mathcal{F}$ denote any field, and let

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*) \tag{164}$$

denote a Leonard system over $\mathcal{F}$. Then there exists a unique sequence of monic polynomials

$$p_0, p_1, \ldots, p_{d+1}; \qquad p_0^*, p_1^*, \ldots, p_{d+1}^*$$

in $\mathcal{F}[\lambda]$ such that

$$\begin{array}{lll} deg\, p_i = i, & deg\, p_i^* = i & (0 \leq i \leq d+1), \\ p_i(A)E_0^* = E_i^* A^i E_0^*, & p_i^*(A^*)E_0 = E_i A^{*i} E_0 & (0 \leq i \leq d), \\ p_{d+1}(A) = 0, & p_{d+1}^*(A^*) = 0. & \end{array}$$

These polynomials satisfy

$$p_0 = 1, \qquad p_0^* = 1, \tag{165}$$

$$\lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \qquad (0 \leq i \leq d), \tag{166}$$

$$\lambda p_i^* = p_{i+1}^* + a_i^* p_i^* + x_i^* p_{i-1}^* \qquad (0 \leq i \leq d), \tag{167}$$

where $x_0,\ x_0^*,\ p_{-1},\ p_{-1}^*$ are all zero, and where

$$\begin{array}{lll} a_i = tr\, E_i^* A, & a_i^* = tr\, E_i A^* & (0 \leq i \leq d), \\ x_i = tr\, E_i^* A E_{i-1}^* A, & x_i^* = tr\, E_i A^* E_{i-1} A^* & (1 \leq i \leq d). \end{array}$$

In fact

$$x_i \neq 0, \qquad x_i^* \neq 0 \qquad (1 \leq i \leq d). \tag{168}$$

We call $p_0, p_1, \ldots, p_{d+1}$ the *monic polynomial sequence* (or *MPS*) of Φ. We call $p_0^*, p_1^*, \ldots, p_{d+1}^*$ the *dual MPS* of Φ. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ, so that

$$\theta_i \neq \theta_j, \qquad \theta_i^* \neq \theta_j^* \qquad \text{if} \quad i \neq j, \qquad (0 \leq i, j \leq d), \tag{169}$$

$$p_{d+1}(\theta_i) = 0, \qquad p_{d+1}^*(\theta_i^*) = 0, \qquad (0 \leq i \leq d). \tag{170}$$

Then

$$p_i(\theta_0) \neq 0, \qquad p_i^*(\theta_0^*) \neq 0 \qquad (0 \leq i \leq d), \tag{171}$$

and

$$\frac{p_i(\theta_j)}{p_i(\theta_0)} = \frac{p_j^*(\theta_i^*)}{p_j^*(\theta_0^*)} \qquad (0 \leq i, j \leq d). \tag{172}$$

Conversely, given polynomials

$$p_0, p_1, \ldots, p_{d+1}; \qquad p_0^*, p_1^*, \ldots, p_{d+1}^* \qquad (174L),\ (174R)$$

in $\mathcal{F}[\lambda]$ satisfying (165)–(168), and given scalars

$$\theta_0, \theta_1, \ldots, \theta_d; \qquad\qquad \theta^*_0, \theta^*_1, \ldots, \theta^*_d \qquad\qquad (175L),\ (175R)$$

in $\mathcal{F}$ satisfying (169)–(172), there exists a Leonard system Φ over $\mathcal{F}$ with MPS (174L), dual MPS (174R), eigenvalue sequence (175L), and dual eigenvalue sequence (175R). The system Φ is unique up to isomorphism of Leonard systems.

In the above paragraph, we described a bijection between the Leonard systems and the systems (174L)–(175R) satisfying (165)–(172). For the case $\mathcal{F} = \mathbb{R}$, the systems (174L)–(175R) satisfying (165)–(172) were previously classified by Leonard [27] and Bannai and Ito [5, p260]. They found the polynomials involved are q-Racah polynomials or related polynomials from the Askey scheme. Their classification has come to be known as Leonard's theorem. Given the above bijection, we may view Theorem 1.9 in the present paper as a "linear algebraic version" of Leonard's theorem. To see one advantage of our version, compare it with the previous version [5, p260]. In that version, the statement of the theorem takes 11 pages. We believe the main value of our version lies in the conceptual simplicity and alternative point of view it provides for the study of orthogonal polynomials.

For the benefit of researchers in special functions we now give more detail on the polynomials that come from Leonard systems. In what follows, we freely use the notation of Definition 1.10. Let Φ denote the Leonard system from (164). In view of (172) we define the polynomials

$$u_i = \frac{p_i}{p_i(\theta_0)} \qquad\qquad (0 \le i \le d),$$

so that

$$u_i(\theta_j) = u^*_j(\theta^*_i) \qquad\qquad (0 \le i, j \le d).$$

The $u_0, u_1, \ldots, u_d$ satisfy a recurrence similar to (166), as we now explain. There exists a unique sequence of scalars $c_1, c_2, \ldots, c_d$; $b_0, b_1, \ldots, b_{d-1}$ taken from $\mathcal{F}$ such that

$$\begin{aligned} x_i &= b_{i-1}c_i && (1 \le i \le d), \\ \theta_0 &= c_i + a_i + b_i && (0 \le i \le d), \end{aligned}$$

where $c_0 = 0, b_d = 0$. Then

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad\qquad (0 \le i \le d-1),$$

and $\lambda u_d - c_d u_{d-1} - a_d u_d$ vanishes on each of $\theta_0, \theta_1, \ldots, \theta_d$.

The polynomials p_i and u_i both satisfy orthogonality relations. Set

$$m_i := tr\, E_i E^*_0 \qquad\qquad (0 \le i \le d).$$

Then each of $m_0, m_1, \ldots, m_d$ is nonzero, and the orthogonality for the p_i is

$$\begin{aligned} \sum_{r=0}^{d} p_i(\theta_r) p_j(\theta_r) m_r &= \delta_{ij} x_1 x_2 \cdots x_i && (0 \le i, j \le d), \\ \sum_{i=0}^{d} \frac{p_i(\theta_r) p_i(\theta_s)}{x_1 x_2 \cdots x_i} &= \delta_{rs} m_r^{-1} && (0 \le r, s \le d). \end{aligned}$$

We remark that since the ground field $\mathcal{F}$ is arbitrary, the question of whether the m_i are positive or not does not arise.

Turning to the u_i, observe that $m_0 = m_0^*$; let n denote the multiplicative inverse of this common value, and set

$$k_i := m_i^* n \qquad\qquad (0 \le i \le d).$$

The orthogonality for the u_i is

$$\begin{aligned} \sum_{r=0}^{d} u_i(\theta_r) u_j(\theta_r) m_r &= \delta_{ij} k_i^{-1} \qquad\qquad (0 \le i, j \le d), \\ \sum_{i=0}^{d} u_i(\theta_r) u_i(\theta_s) k_i &= \delta_{rs} m_r^{-1} \qquad\qquad (0 \le r, s \le d). \end{aligned}$$

We remark

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad\qquad (0 \le i \le d)$$

and

$$n = k_0 + k_1 + \cdots + k_d.$$

All the polynomials and scalars we have described in this section are given by fairly simple rational expressions involving the eigenvalues, dual eigenvalues, φ-sequence and ϕ-sequence of Φ. For example the a_i, a_i^* are given in Lemma 5.1. Using the notation of Definition 4.3, we have

$$b_i = \varphi_{i+1} \frac{\tau_i^*(\theta_i^*)}{\tau_{i+1}^*(\theta_{i+1}^*)}, \qquad\qquad b_i^* = \varphi_{i+1} \frac{\tau_i(\theta_i)}{\tau_{i+1}(\theta_{i+1})}$$

for $0 \le i \le d-1$,

$$c_i = \phi_i \frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i+1}^*(\theta_{i-1}^*)}, \qquad\qquad c_i^* = \phi_{d-i+1} \frac{\eta_{d-i}(\theta_i)}{\eta_{d-i+1}(\theta_{i-1})}$$

for $1 \le i \le d$, and

$$n = \frac{\eta_d(\theta_0)\eta_d^*(\theta_0^*)}{\phi_1 \phi_2 \cdots \phi_d}.$$

Moreover

$$p_i = \sum_{h=0}^{i} \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_1 \varphi_2 \cdots \varphi_h} \frac{\tau_h^*(\theta_i^*)}{\tau_i^*(\theta_i^*)} \tau_h, \qquad\qquad p_i^* = \sum_{h=0}^{i} \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_1 \varphi_2 \cdots \varphi_h} \frac{\tau_h(\theta_i)}{\tau_i(\theta_i)} \tau_h^*,$$

$$u_i = \sum_{h=0}^{i} \frac{\tau_h^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_h} \tau_h, \qquad\qquad u_i^* = \sum_{h=0}^{i} \frac{\tau_h(\theta_i)}{\varphi_1 \varphi_2 \cdots \varphi_h} \tau_h^* \tag{173}$$

for $0 \le i \le d$. Immediately after Theorem 1.9 in the Introduction, we displayed a parametric solution. We now assume the parameters of Φ are given by this solution, and consider the effect

on the associated polynomials. Using this solution and either equation in (173), we find that for $0 \le i, j \le d$, the common value of $u_i(\theta_j), u_j^*(\theta_i^*)$ is given by

$$\sum_{n=0}^{d} \frac{(q^{-i};q)_n (s^* q^{i+1};q)_n (q^{-j};q)_n (s q^{j+1};q)_n q^n}{(r_1 q;q)_n (r_2 q;q)_n (q^{-d};q)_n (q;q)_n}, \tag{174}$$

where

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \qquad\qquad n = 0,1,2\ldots$$

Observe (174) is the basic hypergeometric series

$${}_4\phi_3\left(\begin{matrix} q^{-i},\ s^* q^{i+1},\ q^{-j},\ s q^{j+1} \\ r_1 q,\ \ r_2 q,\ \ q^{-d} \end{matrix}\ ;\ q,\ q\right).$$

The q-Racah polynomials are defined in [25]. Comparing that definition with the above data, and recalling $r_1 r_2 = s s^* q^{d+1}$, we find the u_i and the u_i^* are q-Racah polynomials.

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